INTRODUCTION

TO THE

MATHEMATICAL THEORY

OF THE

STRESS AND STRAIN OF ELASTIC SOLIDS.

AN ELEMENTARY TREATISE

ON

THE DIFFERENTIAL CALCULUS,

CONTAINING

THE THEORY OF PLANE CURVES.

BY

BENJAMIN WILLIAMSON, D.Sc., D.C.L., F.R.S.

EIGHTH EDITION.

AN ELEMENTARY TREATISE

ON

THE INTEGRAL CALCULUS,

CONTAINING

APPLICATIONS TO PLANE CURVES AND SURFACES,

AND ALSO

A Chapter on the Calculus of Variations.

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BY

BENJAMIN WILLIAMSON, D.Sc., D.C.L., F.R.S.,

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SECOND EDITION.

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STRESS AND STRAIN OF ELASTIC SOLIDS.

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PREFACE.

This book, as its title imports, has been written as an introduction merely to the general mathematical theory of Elasticity, but it is hoped that it is sufficient to enable the Student to understand the mathematical theory of the internal strains and stresses that arise whenever external forces are applied to solid bodies. The distribution of stress in structures is a subject of the greatest practical importance, and the study of it is becoming more and more fully recognized as an essential part of the education of an engineer.

One of the first questions that has to be determined by any writer on Elasticity is that of notation. Unfortunately no generally recognized system of nomenclature has as yet been adopted, such as has a place in nearly every other branch of Mathematical Physics. The consequence has been that each writer has usually adopted a different notation for the constituents of strain as well as for those of stress. That which I have adopted is one which was suggested many years ago by the late Professor Townsend. In it the six constituents of strain are denoted by the letters a, b, c, f, g, h, respectively; while the corresponding stress constituents are

represented by the capital letters A, B, C, F, G, H. Accordingly in this notation the elongation quadric is denoted by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = \text{const.},$$

and the stress quadric by

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = \text{const.}$$

As these quadrics occupy so fundamental a place in the general Theory of Elasticity, the foregoing notation has the great advantage of harmonizing with the generally recognized method of representing the equation of a surface of the second degree, and consequently many of the fundamental properties of a quadric can thus at once be transformed into corresponding properties connecting small strains or stresses in elastic solids. There is, at the same time, no difficulty in the transformation of the notation here adopted into that of any other treatise on the subject.

The Student who has sufficient knowledge of higher Mathematics, and leisure for the study of the more advanced portions of the general Mathematical Theory of Elasticity, is referred to the excellent treatise on the subject by Mr. Love, Fellow of St. John's College, Cambridge.

In conclusion I desire to return my best thanks to Professor Tarleton for his kind assistance in the correction of the proof sheets, and for many valuable suggestions throughout.

TRINITY COLLEGE,

February, 1894.

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PRELIMINARY.

1. **Strain, Stress, Elasticity.**—A body is said to be in its natural state when its several parts are in equilibrium under the action of its internal molecular forces solely. Also whenever in consequence of the action of external force a body receives any alteration of form, it is said to be in a state of *Strain*. The internal forces or stresses resulting from such a state of strain are called also the forces of elasticity of the body; such forces tend to restore the body to its natural or unstrained state. Again, for an elastic body in a *given* state of strain, equilibrium must exist, for each portion of the body, between the external forces which act on that portion, and the internal stresses or forces of elasticity. The external forces which act on any portion of a body may be distinguished into *superficial* forces, *i.e.* stresses acting on its boundary, and *body* forces, *i.e.* external forces acting on each element of the body.

We here denote by elasticity the property by which a body when strained tends to return to its natural state, and which, in the case of a perfectly elastic body, restore it to that state when the forces by which it has been strained cease to act.

The following extract from Clerk Maxwell's *Theory of Heat*, p. 295, will help the student towards forming clear ideas on the nature and on the elastic properties of bodies:—"A body which, when subjected to stress, experiences no strain, would, if it existed, be called a *perfectly rigid body*. There are no such bodies, and this definition is

given only to indicate what is meant by perfect rigidity. A body which, when subjected to a given stress at a given temperature, experiences a strain of a definite amount, which does not increase when the stress is prolonged, and which disappears completely when the stress is removed, is called a *perfectly elastic body*. Now, suppose that stresses of the same kind, but of continuously increasing magnitude, are applied to a body in succession, as long as the body returns to its original form when the stress is removed, it is said to be perfectly elastic. If the form of the body is found to be permanently altered when the stress exceeds a certain value, the body is said to be soft or plastic, and the state of the body when the alteration is just going to take place is called the *limit of perfect elasticity*. If the stress be increased till the body breaks, or gives way altogether, the value of the stress is called the *strength of the body for that kind of stress*. If breaking takes place before there is any permanent alteration of form, the body is said to be *brittle*. If the stress when it is maintained constant causes a strain or a displacement in the body which increases continually with the time, the substance is said to be *viscous*. When this continuous alteration of form is only produced by stresses exceeding a certain value, the substance is called a solid, however soft it may be. When the very smallest stress, if continued long enough, will cause a constantly increasing change of form, the body must be regarded as a *viscous fluid*, however hard it may be. Thus, a tallow candle is much softer than a stick of sealing-wax: but if the candle and the stick of sealing-wax are laid between two horizontal supports, the sealing-wax will in a few weeks in summer bend with its own weight, while the candle remains straight. The candle is therefore a soft solid, and the sealing-wax is a viscous fluid."

2. **Plan of this Treatise.**—The subsequent investigations in this book will be principally confined to what are above called perfectly elastic solids, and our results can be applied to actual substances only so far as their properties approximate to those of this ideal elastic body.

When a body is in a condition of stress the mutual distances between its several molecules are in general different from their distances when the body is in its natural

state. Inasmuch as substances expand or contract, in general, under alterations of temperature, we shall in all cases suppose that the temperature of the body is the same after and before the strain. In other words, we shall not in this treatise consider the effects arising from the application of *heat* to elastic bodies.

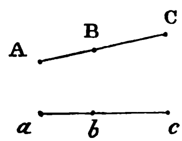
It is evident that we may treat of the state of strain of a body quite independently of its corresponding state of stress. In this point of view the discussion of strain may be regarded as one of pure geometry and analysis, and to this we shall first direct our attention, commencing with the case of homogeneous strain.

CHAPTER I.

STRAIN.

SECTION I.—*Homogeneous Strain.*

3. **Homogeneous Strain.**—A strain is said to be homogeneous when parallel distances in the body continue parallel after the strain, and receive proportionate alterations of length, this alteration of length varying in general with the direction. Thus, suppose A, B, C to be the positions of three particles in the same straight line before the strain, and a, b, c their positions after the strain, then for a homogeneous strain, the points a, b, c must be *in directum*, and we must have

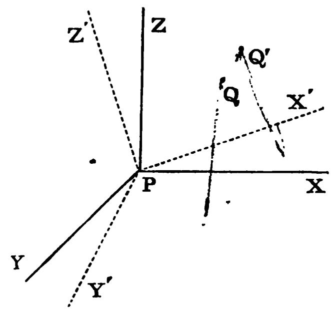


$$\frac{AB}{ab} = \frac{BC}{bc}.$$

It is obvious that in this case a parallelogram remains a parallelogram after the strain, and a parallelepiped remains a parallelepiped.

4. **Elementary forms of Homogeneous Strain.**—Some of the more elementary forms of strain may be here specified, as follows:—(1) *Spherical Strain.*—In this case all distances are altered by the strain in the same constant ratio; and accordingly all portions of the body are of the same shape after as before the strain. (2) *Cylindrical strain.*—In this case all distances in one direction are unaltered in length, while distances perpendicular to that direction are uniformly lengthened or contracted. (3) *Stretch.*—In this case all distances in one direction are uniformly altered in length, while all distances perpendicular to that direction are unaltered in length. This is sometimes called a *simple elongation.*

5. **Geometrical Representation of Homogeneous Strain.**—It is obvious that no strain is produced by a motion of translation of the body, or by a rotation round any line. Hence, after the strain, we may suppose any particle P of the body brought back to its original position, the other particles receiving the same motion of translation as that given to P. Accordingly, we may regard



P as fixed, and that PX, PY, PZ are any three rectangular lines drawn in the body before the strain. Suppose PX', PY', PZ' to be the positions of the lines PX, PY, PZ, respectively, after the strain. Again, let Q be the position of any particle of the body before, and Q' its position after the strain; and let xyz be the coordinates of Q referred to the axes PX, PY, PZ; and $x'y'z'$ those of Q' referred to PX', PY', PZ'.

As the strain is homogeneous the ratios $\frac{x'}{x}$, $\frac{y'}{y}$, $\frac{z'}{z}$ are each constant, and are the same for all points of the body.

Accordingly we may write

$$x' = ax, \quad y' = by, \quad z' = cz, \tag{1}$$

where a, b, c are constants for the same homogeneous strain.

From these equations it follows that all points in a plane before strain lie in a new plane after the strain; also that points on any surface of the second degree continue after the strain to lie on a surface of the second degree, &c.

6. **Strain Ellipsoid.**—From the preceding it follows that any sphere becomes, in general, an ellipsoid after the strain; and hence points on a sphere with radius unity and P as centre lie on the ellipsoid

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1, \tag{2}$$

after the strain.

This is called the strain ellipsoid for the point P; and from the form of its equation we see that the axes PX', PY', PZ' are a system of conjugate diameters of this ellipsoid.

Accordingly, any system of rectangular lines become, after the strain, a system of conjugate diameters of the strain ellipsoid.

Consequently, there is in general one, and but one, rectangular system of lines which remain rectangular after the strain; viz. the axes of the strain ellipsoid. These are called the *principal axes of the strain.*

7. **Pure Strain.**—Whenever the axes of the strain ellipsoid are coincident with their original directions the strain is said to be pure. Moreover, as any system of rectangular lines can, by a rotation, be made to coincide with any other rectangular system having the same vertex, it follows that

Any homogeneous strain is equivalent to a pure strain combined with a rotation.

8. **Elongation.—Principal Elongations.** The elongation of any strained line is the ratio of its increment of length to its original length.

Thus, in Art. 3, the elongation e of the distance AB is given by

$$e = \frac{ab - AB}{AB}.$$

This gives

$$ab = AB\,(1 + e). \tag{3}$$

Here, as in all similar cases, to a decrease of length corresponds a *negative elongation.*

Again, in Art. 5, the elongations corresponding to the axes PX, PY, PZ are $a - 1$, $b - 1$, $c - 1$, respectively.

The elongations for the axes of the strain ellipsoid are called the *principal elongations* of the strain, and it is immediately seen that the elongations corresponding to the greatest and least axes of this ellipsoid are the greatest and the least elongations of the strain.

EXAMPLES.

1. Prove that the ratio of any two areas situated in the same plane, or in parallel planes, is unaltered by a homogeneous strain.

Let $ABCD$ and $A'B'C'D'$ be two parallelograms, such that AB and $A'B'$ are parallel, and AC and $A'C'$ also parallel; also let $abcd$, and $a'b'c'd'$ be the corresponding parallelograms after strain.

Then, if e and e' be the corresponding elongations, we have, by (3),

$$ab = AB(1+e), \quad ac = AC(1+e'), \quad a'b' = A'B'(1+e), \quad a'c' = A'C'(1+e').$$

Also, area $\quad ABCD : A'B'C'D' = AB \times AC : A'B' \times A'C'$

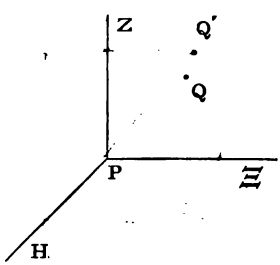
$$\doteq ab \times ac : a'b' \times a'c' = \text{area } abcd : \text{area } a'b'c'd'.$$

Again, any area can be divided by indefinitely near parallel lines into an indefinite number of strips, which are approximately parallelograms; and we easily infer from the above that the ratio of two such areas for parallel planes is unaltered by the strain.

2. Prove the same property for any two volumes.

As in Ex. 1, we readily see that the ratio of the volumes of any two parallelepipeds, whose edges are respectively parallel, is unaltered by a homogeneous strain; hence the result follows, for any volume can be ultimately divided into a number of indefinitely small parallelepipeds whose edges are respectively parallel to any three given directions.

9. **Equations for a Homogeneous Strain.** — We suppose the strain referred to a system of coordinate axes, as in Art. 8. In doing so, however, we will find it more convenient to alter the notation, and to denote by $\xi\eta\zeta$ the coordinates of Q; and by $\xi'\eta'\zeta'$ the coordinates of Q', referred to the *same axes*.



From (1) we may obviously write

$$\xi' = a_1\xi + a_2\eta + a_3\zeta, \quad \eta' = b_1\xi + b_2\eta + b_3\zeta, \quad \zeta' = c_1\xi + c_2\eta + c_3\zeta, \quad (4)$$

in which $a_1a_2a_3 \; b_1b_2b_3 \; c_1c_2c_3$ are constants for the same strain.

Now, let

$$\Delta\xi = \xi' - \xi, \quad \Delta\eta = \eta' - \eta, \quad \Delta\zeta = \zeta' - \zeta; \qquad (5)$$

then $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ are the coordinates of Q' relative to a parallel system of axes drawn through Q; i. e. $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ are the components of the *displacement* of Q arising from the strain.

Hence we get

$$\left.\begin{aligned}\Delta\xi &= (a_1 - 1)\,\xi + a_2\eta + a_3\zeta \\ \Delta\eta &= b_1\xi + (b_2 - 1)\,\eta + b_3\zeta \\ \Delta\zeta &= c_1\xi + c_2\eta + (c_3 - 1)\,\zeta\end{aligned}\right\}. \qquad (6)$$

Let $\qquad PQ = \rho,\ PQ' = \rho + \Delta\rho;$

then

$$(\rho + \Delta\rho)^2 = (\xi + \Delta\xi)^2 + (\eta + \Delta\eta)^2 + (\zeta + \Delta\zeta)^2;$$

hence, if the displacements $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ be very small in comparison with ξ, η, ζ, we have, approximately,

$$\rho\Delta\rho = \xi\Delta\xi + \eta\Delta\eta + \zeta\Delta\zeta = (a_1 - 1)\,\xi^2 + (b_2 - 1)\eta^2$$
$$+ (c_3 - 1)\,\zeta^2 + (a_2 + b_1)\,\xi\eta + (a_3 + c_1)\,\zeta\xi + (b_3 + c_2)\,\eta\zeta. \quad (7)$$

Now, let

$$\left.\begin{aligned}a_1 - 1 = a,\quad b_2 - 1 = b,\quad c_3 - 1 = c, \\ 2f = c_2 + b_3,\quad 2g = a_3 + c_1,\quad 2h = b_1 + a_2\end{aligned}\right\}, \qquad (8)$$

and we may write, in the case of a *small strain*,

$$\rho\Delta\rho = a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta. \qquad (9)$$

It may be here observed that, unless the contrary be stated, all our subsequent investigations will be restricted to the case of small strains.

Example.

1. Show that there is in general one line in the body which remains unaltered in direction by the strain.

If this be so we should have

$$\frac{\xi'}{\xi} = \frac{\eta'}{\eta} = \frac{\zeta'}{\zeta} = \lambda \text{ (suppose).}$$

Hence equations (4) give the following cubic for λ:—

$$\begin{vmatrix} a_1 - \lambda, & a_2, & a_3 \\ b_1, & b_2 - \lambda, & b_3 \\ c_1, & c_2, & c_3 - \lambda \end{vmatrix} = 0.$$

Now, every cubic has at least one real root: accordingly there is always one direction unaltered by the strain.

It may be observed that this result is true for all strains, whether small or not.

10. **Elongation Quadric.**—Suppose the point Q to lie on the quadric U, represented by the equation

$$U \equiv a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\zeta\eta = K, \qquad (10)$$

where K is a small constant quantity, then from (9) we get

$$\frac{\Delta\rho}{\rho} = \frac{K}{\rho^2}. \qquad (11)$$

Again, by Art. 8, $\frac{\Delta\rho}{\rho}$ represents the elongation of PQ; and we see that, in *small strains*, the elongation of any radius vector of the quadric U varies inversely as the square of the vector.

From this property the quadric (10) is called the elongation quadric of the strain. It may be regarded as a *fixed quadric drawn in the body;* and by means of it the elongation in any direction can be found when $abcfgh$ are known.

From its equation it is evident that the elongation quadric is a central surface, having P as centre. It is an ellipsoid or an hyperboloid, according to the relative magnitudes of constants a, b, c, &c.

If the elongation quadric is an ellipsoid, then, by (10), all lines are elongated if K be positive; and are contracted if K be negative.

For an hyperboloid, the surfaces represented by

$$\left.\begin{aligned} a\xi^2 + b\eta^2 + c\zeta^2 + 2f\zeta + 2\zeta\xi + 2h\xi\eta &= +K \\ a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta &= -K \end{aligned}\right\}. \qquad (12)$$

are called conjugate hyperboloids; and we see that all central vectors which meet the former are elongated, while those that meet the latter are contracted.

Along the asymptotic cone

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0\,; \qquad (13)$$

the lines are unaltered in length, but may of course be altered in direction.

Again, if e be the elongation in any direction, we have, from (11),

$$e = \frac{K}{\rho^2}, \text{ or } K = e\,(\xi^2 + \eta^2 + \zeta^2).$$

Consequently lines of the same elongation e lie on the cone.

$$(a-e)\xi^2+(b-e)\eta^2+(c-e)\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta=0. \quad (14)$$

Again, if lmn be the direction cosines of ρ, we have for the corresponding elongation e,

$$e=al^2+bm^2+cn^2+2fmn+2gnl+2hlm. \quad (15)$$

If we suppose the equation of the elongation quadric when referred to its principal axes to be

$$e_1\xi^2+e_2\eta^2+e_3\zeta^2=K, \quad (16)$$

then it follows from (9) that e_1, e_2, e_3 are the elongations along the axes of the quadric, and are accordingly the principal elongations of the strain.

If the asymptotic cone

$$e_1\xi^2+e_2\eta^2+e_3\zeta^2=0$$

breaks up into two planes, there will be two *planes of no distortion*. This takes place when one of the principal elongations vanishes, and the two others have opposite signs.

11. **Constituents of a Small Strain.**—From the properties of the elongation quadric it follows that a small homogeneous strain is completely determined when the constants $abcfgh$ are known, for by means of these the strain in any direction can be at once determined by (15).

Accordingly, $abcfgh$ are called the six constituents of the strain.

It is plain from (8) that a, b, c are the elongations for the coordinate axes. In the next article we shall give the geometrical interpretation of f, g, h for a small strain.

It may be observed that all strains in natural bodies are in general very small; and accordingly, in all practical questions, it is sufficient to restrict our investigations to small strains.

If a body first receive the small strain represented by $abcfgh$, and afterwards receives the small strain $a'b'c'f'g'h'$,

then the whole strain is equivalent to the single strain represented by

$$a + a', \ b + b', \ c + c', f + f', \ g + g', \ h + h'.$$

Thus we can readily infer that any number of small strains can be added or subtracted, so as to give the resultant strain.

12. **Geometrical Interpretation of the quantities** f, g, h. **Pure Shears.**—If, in Art. 9, we suppose Q taken on the axis of Ξ, equations (4) become

$$\xi' = a_1\xi, \quad \eta' = b_1\xi, \quad \zeta' = c_1\xi.$$

Hence if l_1, m_1, n_1 be the direction cosines of PQ', the strained direction of $P\Xi$, we have

$$l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \ m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \ n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}.$$

Likewise if l_2, m_2, n_2 correspond to the strained position of the axis of η, we have

$$l_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \ m_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \ n_2 = \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Hence, if ϕ_3 be the angle between the strained positions of the axes of ξ and η, we have

$$\cos\phi_3 = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^3}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{(1 + a)\,a_2 + (1 + b)\,b_1 + c_1c_2}{\sqrt{(1 + a)^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + (1 + b)^2 + c_2^2}}, \text{ by (8).}$$

Accordingly, for a small strain, *neglecting small quantities of the second order*, we get

$$\cos\phi_3 = a_2 + b_1 = 2h. \qquad (17)$$

In like manner

$$\cos\phi_2 = 2g, \quad \cos\phi_1 = 2f,$$

where ϕ_2, ϕ_1 have similar meaning.

Again, since the angles between the strained directions of the coordinate axes are nearly right angles, we see that

$$2h = \frac{\pi}{2} - \phi_3, \quad 2g = \frac{\pi}{2} - \phi_2, \quad 2f = \frac{\pi}{2} - \phi_1, \tag{18}$$

and consequently $2f$, $2g$, $2h$ are the small angular changes produced by the strain in the coordinate system of axes.

It should be observed that when h is *positive* the angle between the positive directions of the axes of ξ and η is *diminished* by the strain, and similarly for g and h. The quantities $2f$, $2g$, $2h$ are called *pure shears*, since their effects, as shown above, consist in angular, or what is called *shearing* distortion.

Examples.

1. If in the unstrained body a unit cube whose edges are parallel to the coordinate axes be taken, and if we take a corresponding parallelepiped, whose edges are all of unit length and are those of the cube after the strain; show that the elongations of the diagonals of the faces of this parallelepiped are respectively equal to f, g, h.

2. Show that a shearing strain can be produced by a small elongation along one right line, combined with an equal compression along a perpendicular line. This, from the elongation quadric, is equivalent to showing that the curve

$$2h\xi\eta = K$$

when referred to its axes has for its equation

$$h(\xi_1^2 - \eta_1^2) = K.$$

3. Find, with reference to any three rectangular axes, the strain constituents arising from a simple elongation e in a given direction.

Take this direction as axis of x', and let l, m, n be its direction cosines. Then the equation of the elongation quadric is $ex'^2 = K$, and therefore its equation referred to the given axes is

$$e(lx + my + nz)^2 = K.$$

Hence the constituents in question are

$$el^2, \quad em^2, \quad en^2, \quad elm, \quad eln, \quad emn.$$

4. Find the conditions among the principal strains that a given strain shall be reducible to a pure shear.

The equation of the elongation quadric in this case is reducible to

$$2h\xi\eta = K.$$

Hence one of the principal strains must be zero, and the other two must be equal and have opposite signs.

5. Show that if the strain be a spherical strain we must have

$$a = b = c, \text{ and } f = g = h = 0.$$

13. **Equation to Strain Ellipsoid.**—Let equations (4) be written

$$\left.\begin{aligned}\xi' &= (1+a)\,\xi + a_2\eta + a_3\zeta\\ \eta' &= b_1\xi + (1+b)\,\eta + b_3\zeta\\ \zeta' &= c_1\xi + c_2\eta + (1+c)\,\zeta\end{aligned}\right\}. \qquad (19)$$

Then, since a_1, a_2, &c., are, by hypothesis, very small, we see that when small quantities of the second order are neglected, we may substitute ξ' for ξ, η' for η, and ζ' for ζ in the small terms in (19), so that these equations lead to

$$\begin{aligned}\xi &= (1-a)\,\xi' - a_2\eta' - a_3\zeta',\\ \eta &= (1-b)\,\eta' - b_1\xi' - b_3\zeta',\\ \zeta &= (1-c)\,\zeta' - c_1\xi' - c_2\eta'.\end{aligned}$$

If now Q lies on the sphere

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

then will Q' lie on the ellipsoid

$$\begin{aligned}\{(1-a)\,\xi' - a_2\eta' - a_3\zeta'\}^2 + \{(1-b)\,\eta' - b_1\xi' - b_3\zeta'\}^2&\\ + \{(1-c)\zeta' - c_1\xi' - c_2\eta'\}^2 = 1.&\end{aligned}$$

If in this equation small quantities of the second order be omitted, and if the accents be dropped for convenience, then will Q' lie on the ellipsoid

$$\xi^2 + \eta^2 + \zeta^2 - 2\,(a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta) = 1. \qquad (20)$$

This is the equation to the strain ellipsoid of Art. 6; from its form it is evident that the strain ellipsoid and the elongation quadric have codirectional axes, and also have the same circular sections. It is also obvious that along these circular sections the dilatation is uniform, and that the strain in these planes is a pure dilatation without distortion. Also the elongation for any line in either of these planes is equal to that for the mean axis of the elongation quadric.

Again, from (16) the equation of the strain ellipsoid when referred to its axes becomes

$$\xi^2(1-2e_1) + \eta^2(1-2e_2) + \zeta^2(1-2e_3) = 1. \qquad (21)$$

14. **Cubical Dilatation of the Strain.**—We have shown in Ex. 2, Art. 8, that the ratio of any volume to the corresponding volume after the strain is constant; we now proceed to determine this ratio for the case of small strains.

Suppose any parallelepiped taken, whose edges are respectively parallel to the coordinate axes, and let l, m, n be the lengths of its edges: then after the strain the lengths of the edges of the strained parallelepiped are

$$l(1+a), \quad m(1+b), \quad n(1+c).$$

Again, since by hypothesis, the shears fgh are very small, the strained parallelepiped is nearly rectangular, and, accordingly, neglecting small quantities of the second order, the sines of the angles between its edges are each unity; hence, if V and V' be the respective volumes of the parallelepipeds, we have

$$\begin{aligned} V' &= lmn\,(1+a)(1+b)(1+c) \\ &= V(1+a)(1+b)(1+c) \\ &= V(1+a+b+c), \end{aligned}$$

neglecting small quantities of the second order.

If Δ denote the cubical dilatation, we have

$$\Delta = \frac{V'-V}{V} = a+b+c. \qquad (22)$$

Again, when $\Delta = 0$, the strain is called an incompressible strain.

15. **General Equations of Strain.**—If now we make the following additional transformations

$$c_2 - b_3 = 2\theta_1, \quad a_3 - c_1 = 2\theta_2, \quad b_1 - a_2 = 2\theta_3. \qquad (23)$$

Equations (6) may be written

$$\left.\begin{aligned} \Delta\xi &= a\xi + h\eta + g\zeta + \theta_2\zeta - \theta_3\eta \\ \Delta\eta &= h\xi + b\eta + f\zeta + \theta_3\xi - \theta_1\zeta \\ \Delta\zeta &= g\xi + f\eta + c\zeta + \theta_1\eta - \theta_2\xi \end{aligned}\right\}. \qquad (24)$$

In these expressions the terms $\theta_2\zeta - \theta_3\eta$, $\theta_3\xi - \theta_1\eta$, $\theta_1\eta - \theta_2\xi$ represent the changes in position due to small *angular displacements* θ_1, θ_2, θ_3, made respectively round the coordinate axes (*Dynamics*, Art. 253); also the remaining terms correspond to a pure strain, as will be shown in the next Article. We thus see again that any homogeneous strain can be resolved into a pure strain, combined with a rotation.

16. **Pure Strain.**—We now proceed to a geometrical interpretation of the equations

$$\Delta\xi = a\xi + h\eta + g\zeta, \quad \Delta\eta = h\xi + b\eta + f\zeta, \quad \Delta\zeta = g\xi + f\eta + c\zeta. \qquad (25)$$

Suppose the point $\xi\eta\zeta$, taken on the elongation quadric,

$$U = a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\zeta\eta = K,$$

then

$$\frac{1}{2}\frac{dU}{d\xi} = a\xi + h\eta + g\zeta = \Delta\xi, \text{ \&c.}$$

Hence

$$\Delta\xi : \Delta\eta : \Delta\zeta = \frac{dU}{d\xi} : \frac{dU}{d\eta} : \frac{dU}{d\zeta} = l : m : n, \qquad (26)$$

where l, m, n are the direction cosines of the normal to the quadric U at the point Q.

These equations show that the displacement QQ' is normal in this case to the elongation quadric U. Hence, if the line PQ' coincide in direction with PQ, the line PQ must be normal to the quadric, *i. e.* PQ must be one of its principal axes.

Hence, in this case the principal axes of the elongation quadric undergo no angular displacements by the strain; accordingly, the strain is pure (Art. 7). A pure strain is also called a *nonrotational* strain. It is also obvious that the axes of the quadric U are the principal axes of the strain.

The converse theorem, that if the strain, represented by equations (6), is a pure strain, we must have $a_2 = b_1$, $c_1 = a_3$, $b_3 = c_2$, can be established without difficulty. This is left as an exercise for the student.

17. **Principal Elongations of the Strain.**—We have shown, in Art. 10, that the principal elongations of the strain

correspond to the axes of the elongation quadric. Hence (Salmon's *Geometry of Three Dimensions*, Art. 83), the principal elongations are the roots of the cubic

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix} = 0. \qquad (27)$$

This result can be also readily shown from equation (15), viz.

$$e = al^2 + bm^2 + cn^2 + 2fmn + 2gln + 2hlm.$$

For, if e be a maximum or a minimum, we must have, regarding l, m, n as variables,

$$(al + hm + gn)\,dl + (hl + bm + fn)\,dm + (gl + fm + cn)\,dn = 0\,;$$

also from the equation

$$l^2 + m^2 + n^2 = 1,$$

we get

$$ldl + mdm + ndn = 0.$$

These equations give, immediately,

$$\left.\begin{aligned} al + hm + gn &= \lambda l \\ hl + bm + fn &= \lambda m \\ gl + fm + cn &= \lambda n \end{aligned}\right\}, \qquad (28)$$

where λ is an indeterminate multiplier.

If lmn be eliminated from equations (28) we get, at once, the determinant given in (27).

Again, if in (28) the first equation be multiplied by l, the second by m, and the third by n, we find by addition that $\lambda = e$.

18. **Invariants of the Strain.**—Since the elongation quadric is independent of the directions of the particular system of rectangular coordinates to which it is referred, it follows that the coefficients of λ in (27) are *invariants*;

hence the quantities, $a+b+c$, $ab+bc+ac-f^2-g^2-h^2$, and the determinant

$$D = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}, \tag{29}$$

are unaltered by orthogonal transformations, *i.e.* are invariants of the strain. We have already seen, in Art. 14, that the first of these expressions represents the cubical dilatation of the strain.

A few applications of these invariants are given in the following examples.

EXAMPLES.

1. Find the conditions that a given strain should be equivalent to a simple shear. *Ans.* $a+b+c=0$, and $D=0$.

2. Find the magnitude of the shear s in this case.
Here we obviously have

$$s^2 = f^2+g^2+h^2-ab-ac-bc = f^2+g^2+h^2+\tfrac{1}{2}(a^2+b^2+c^2),$$

since $a+b+c=0$.

3. Find the condition that there should be two planes of no distortion. *Ans.* $D=0$.

19. **Uniplanar Strain.**—In many cases the displacement of every point is parallel to a *fixed* plane. Such a strain is called two-dimensional or uniplanar. In this case we may take the fixed plane as that of $\xi\eta$, and therefore $c=0$, $f=0$, $g=0$; then the constituents of the strain are a, b, and h; and the discussion of the strain is greatly simplified.

For instance the elongation quadric becomes

$$ax^2+by^2+2hxy=K, \tag{30}$$

and is either an elliptic or hyperbolic cylinder. Also the principal elongations are the roots of the quadratic

$$(a-\lambda)(b-\lambda)-h^2=0. \tag{31}$$

Examples.

1. In a given plane find the pair of rectangular lines—(1) for which the shear is evanescent; (2) for which the shear is a maximum.

Taking the given plane for that of $\xi\eta$, the corresponding section of the stress quadric is represented by

$$a\xi^2 + b\eta^2 + 2h\xi\eta = K. \qquad a)$$

Transform to new axes in the same plane, and let a', b', h' be the corresponding coefficients in the transformed expression; then, by a well-known elementary result we have $ab - h^2 = a'b' - h'^2$, and $a + b = a' + b'$.
Hence we have

$$4h'^2 + (a' - b')^2 = 4h^2 + (a - b)^2.$$

(1) It is obvious that $h' = 0$ when the new axes are the axes of the conic (a).
(2) Again, h' is a maximum when $a' = b'$, *i.e.* for the lines which bisect the angles between the axes of the conic (a): also this maximum value is

$$\tfrac{1}{2}\sqrt{4h^2 + (a - b)^2}, \text{ that is } = \tfrac{1}{2}(a_1 - b_1),$$

where

$$a_1\xi_1^2 + b_1\eta_1^2 = K$$

is the equation of the conic (a) when referred to its axes.

2. In any small homogeneous strain, if OP, OQ, OR be a rectangular system of lines drawn before the strain, prove that, however the rectangular lines OQ and OR be varied, OP remaining fixed, the sum of the squares of the shears of the angles POQ and POR is constant.

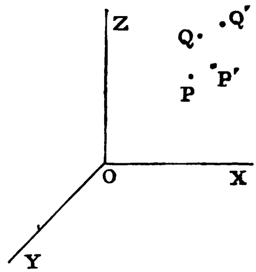
Taking OP for axis of ξ, we see immediately that $f^2 + g^2$ remains constant, however the other rectangular axes be altered.

3. In a pure homogeneous strain, prove the existence of three rectangular axes of pure radial dilatation, unaccompanied by change of direction, and also of two planes of pure areal dilatation, unaccompanied by distortion of form.

4. Show geometrically that the most general strain at any point of an elastic solid is equivalent to a pure strain together with a rotation. Also show that a strain can in all cases be resolved into—(a) a spherical dilatation; (b) a simple elongation; together with (c) a pure shear.

SECTION II.—*Heterogeneous Strain.*

20. **Expressions for a Heterogeneous Strain.** We now suppose the state of strain to vary from point to point in the body, but in doing so will assume that the strains are continuous quantities.



Suppose the particles of the body referred to a rectangular system of axes OX, OY, OZ. Let xyz be the coordinates of any particle P before the strain, and $x+u$, $y+v$, $z+w$, the coordinates of P', the position of the particle after the strain; then u, v, w are the components of the displacement of P caused by the strain.

Now, in any strain we suppose that the displacements of any particle depend on its position, *i. e.* on its coordinates. Accordingly we may assume that

$$u = \phi_1(x, y, z), \quad v = \phi_2(x, y, z), \quad w = \phi_3(x, y, z), \qquad (1)$$

where the form of the functions depend on the character of the strain.

Again, let $x+\xi$, $y+\eta$, $z+\zeta$, be the coordinates, before the strain, of any particle Q, near to P: and let Q' be the strained position of Q: also let u', v', w' be the displacements of Q, and ξ', η', ζ' the coordinates of Q' *relative* to P'; then it is plain that we have

$$\xi' - \xi = u' - u, \quad \eta' - \eta = v - v', \quad \zeta' - \zeta = w' - w. \qquad (2)$$

Moreover, by (1), we have

$$u' = \phi_1(x+\xi, \ y+\eta, \ z+\zeta);$$

therefore

$$\xi' - \xi = u' - u = \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz},$$

neglecting small quantities of the second order.

Hence we have, approximately,

$$\left.\begin{aligned} \xi' &= \xi\left(1+\frac{du}{dx}\right) + \eta\frac{du}{dy} + \zeta\frac{du}{dz} \\ \eta' &= \xi\frac{dv}{dx} + \eta\left(1+\frac{dv}{dy}\right) + \zeta\frac{dv}{dz} \\ \zeta' &= \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\left(1+\frac{dw}{dz}\right) \end{aligned}\right\}. \qquad (3)$$

Now, these are the same as equations (4), Art. 9, provided we have

$$\left.\begin{aligned} a_1 &= 1+\frac{du}{dx}, & a_2 &= \frac{du}{dy}, & a_3 &= \frac{du}{dz}, \\ b_1 &= \frac{dv}{dx}, & b_2 &= 1+\frac{dv}{dy}, & b_3 &= \frac{dv}{dz}, \\ c_1 &= \frac{dw}{dx}, & c_2 &= \frac{dw}{dy}, & c_3 &= 1+\frac{dw}{dz} \end{aligned}\right\}. \qquad (4)$$

Also, since $\frac{du}{dx}, \frac{du}{dy}$, &c., are, by hypothesis, *continuous* functions, we may, *neglecting small quantities of the second order*, consider them as constant for points in the immediate neighbourhood of P; accordingly, we can immediately infer that *all the results previously established for small homogeneous strain hold good equally for heterogeneous strain, on making the above substitutions* for a_1, a_2, a_3, b_1, &c.

21. Constituents of the Strain.—The strain in the immediate neighbourhood of any point P is completely determined when the strain quadric of the point is known, *i. e.* when the quantities

$$\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}, \tfrac{1}{2}\left(\frac{du}{dy}+\frac{dv}{dx}\right), \tfrac{1}{2}\left(\frac{dw}{dx}+\frac{du}{dz}\right), \tfrac{1}{2}\left(\frac{dv}{dz}+\frac{dw}{dy}\right),$$

are known. These six quantities are called the constituents of the strain, as in Art. 11; and we shall continue to denote them by the letters *abcfgh*, as before, but will suppose that they vary in general from point to point in the body.

Also, by Art. 14, the cubical dilatation at any point is given by the equation

$$\Delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}. \qquad (5)$$

22. **Pure Strain.—Strain Potential — Displacement Lines**—From Art. 16 we see that the strain is a pure strain when we have

$$\frac{du}{dy} = \frac{dv}{dx}, \quad \frac{dw}{dx} = \frac{du}{dz}, \quad \frac{dv}{dz} = \frac{dw}{dy}. \qquad (6)$$

Accordingly, in a pure strain,

$$udx + vdy + wdz$$

must be an exact differential.

Hence, in this case, we may write

$$udx + vdy + wdz = d\phi, \qquad (7)$$

where ϕ is some function of the coordinates xyz.

The function ϕ is called the potential of the pure strain, and the constituents of the strain are

$$\frac{d^2\phi}{dx^2}, \quad \frac{d^2\phi}{dy^2}, \quad \frac{d^2\phi}{dz^2}, \quad \frac{d^2\phi}{dx\,dy}, \quad \frac{d^2\phi}{dz\,dx}, \quad \frac{d^2\phi}{dy\,dz}.$$

Again, the cubical dilatation of a pure strain is expressed by

$$\Delta = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = \nabla^2\phi. \qquad (8)$$

For incompressible substances, we have

$$\nabla^2\phi = 0 \qquad (9)$$

in the case of a pure strain.

From the preceding it is readily seen that the function ϕ possesses properties analogous to those of the potential function in Attractions.

Thus the surface

$$\phi = \text{const.}$$

is called an equipotential surface of the strain: also, since

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz}, \tag{10}$$

it follows that the displacement of any point is normal to the equipotential surface passing through the point.

Again, since in general the direction of the displacement satisfies the equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \tag{11}$$

it follows that these are the differential equations of a series of curved lines, called *displacement lines;* these lines correspond to lines of force in Attractions.

Again, to a tube of force in Attractions corresponds *a displacement tube*, &c.

23. **Application to Uniplanar Incompressible Strain.**—If the strain be uniplanar the displacement curves satisfy the equation

$$udy - vdx = 0. \tag{12}$$

Moreover, if the strain be incompressible, we have

$$\frac{du}{dx} + \frac{dv}{dy} = 0.$$

But this is the condition that $udy - vdx$ should be an exact differential: accordingly, for an incompressible strain, we may assume that

$$udy - vdx = d\psi, \tag{13}$$

where ψ is some function of x and y.

Hence we have

$$u = \frac{d\psi}{dy}, \quad v = -\frac{d\psi}{dx}, \tag{14}$$

where ψ = constant is the equation of a displacement line.

Again, if the strain be *pure*, we have also

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy};$$

and consequently in that case,

$$\frac{d\phi}{dx} = \frac{d\psi}{dy}, \quad \frac{d\phi}{dy} = -\frac{d\psi}{dx}. \tag{15}$$

The functions ϕ and ψ obviously satisfy the differential equations

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0, \quad \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0. \tag{16}$$

Also since

$$\frac{d\phi}{dx}\frac{d\psi}{dx} + \frac{d\phi}{dy}\frac{d\psi}{dy} = 0,$$

it follows that the curves represented by ϕ = const. and ψ = const. form two systems of mutually orthogonal curves. The system of curves represented by ϕ = const. are called *equipotential curves*, and those represented by ψ = const. are the corresponding *displacement curves*.

24. **Conjugate Functions.**—If any function of $x + iy$, where $i = \sqrt{-1}$, be represented by $f(x + iy)$, and if we suppose

$$f(x+iy) = X + iY, \tag{17}$$

where X and Y are real functions of x and y, then it is immediately seen, by differentiation, that we must have

$$\frac{dX}{dx} = \frac{dY}{dy}, \text{ and } \frac{dX}{dy} = -\frac{dY}{dx}. \tag{18}$$

Any functions X and Y, which satisfy (17), are called *conjugate functions*; and accordingly, all functions which are connected by equations (18) are conjugate to each other. Hence, we see that the potential and displacement functions ϕ and ψ of Art. 23 are conjugate to each other.

We also infer that any pair of conjugate functions may be regarded as capable of representing a possible incompressible pure strain for an elastic solid. Some applications of this statement will be found among the examples at the end of the chapter.

25. **Vortex Lines, and Vortex Tubes.**—When the strain is rotational, then, in addition to the strain proper, each element receives a small angular displacement, called a *vortical rotation*, of which the three components, by Art. 15, are

$$\theta_1 = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),\ \theta_2 = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right),\ \theta_3 = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right). \quad (19)$$

If at each point an indefinitely short line be drawn in the direction of the axis of the corresponding rotation, it will satisfy the equations

$$\frac{dx}{\theta_1} = \frac{dy}{\theta_2} = \frac{dz}{\theta_3}. \qquad (20)$$

These may be regarded therefore as the differential equations of a system of curves called *vortex lines*.

Also, in analogy with displacement tubes, we may regard a system of vortex lines drawn through all the points on the boundary of any small area; such a system of lines form what is called a *vortex tube*.

Since the strain at any point is completely determined when the elongation quadric is known, we see that the vortical rotation here considered is not properly a part of the strain. This will also appear more exactly when we consider the stresses which correspond to a given state of strain for a perfectly elastic solid; for it will be shown that the stresses at any point can be always expressed in terms of the quantities here designated by *abcfgh*, and do not depend at all on the vortex displacements.

26. **Spherical Pure Strain.**—In many cases the displacement of each point P of the body takes place along the line OP which joins the point to a fixed point O.

If O be taken as the origin of a rectangular system of coordinates, then for any point P we may write

$$u = x\psi,\ v = y\psi,\ w = z\psi. \qquad (21)$$

Hence

$$udx + vdy + wdz = (xdx + ydy + zdz)\psi = r\psi\, dr. \qquad (22)$$

Accordingly, for a *pure strain*, we have

$$d\phi = r\psi\, dr, \text{ or } \psi = \frac{1}{r}\frac{d\phi}{dr}. \tag{23}$$

Hence, the normal constituents of the strain are given by

$$\left.\begin{aligned} a &= \frac{du}{dx} = \psi + \frac{x^2}{r}\frac{d\psi}{dr}, \\ b &= \psi + \frac{y^2}{r}\frac{d\psi}{dr}, \quad c = \psi + \frac{z^2}{r}\frac{d\psi}{dr} \end{aligned}\right\}. \tag{24}$$

Again,

$$\Delta = a + b + c = 3\psi + r\frac{d\psi}{dr} = \frac{1}{r^2}\frac{d}{dr}(r^3\psi) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right). \tag{25}$$

Also

$$\left.\begin{aligned} f &= \tfrac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right) = \frac{yz}{r}\frac{d\psi}{dr}, \\ g &= \frac{zx}{r}\frac{d\psi}{dr}, \quad h = \frac{xy}{r}\frac{d\psi}{dr} \end{aligned}\right\}. \tag{26}$$

In order to determine the *principal strains* at any point P, suppose OP taken on the axis of x, then the coordinates of P become

$$x = r,\ y = z = 0,$$

and we have

$$f = g = h = 0.$$

Hence, if we denote the strain along OP by e_1, and the perpendicular strains by e_2 and e_3, we see at once from (15), that

$$e_1 = \psi + r\frac{d\psi}{dr} = \frac{d^2\phi}{dr^2},\ e_2 = e_3 = \psi = \frac{1}{r}\frac{d\phi}{dr}. \tag{27}$$

In this case the equipotential surfaces are a system of spheres, having O as a common centre; and all points lying before the strain on one of these spheres lie after the strain on a concentric sphere. The strain may accordingly be called a spherical strain; however, in this case the condensation Δ is not constant, as in Art. 5, but varies with the distance r.

27. **Cylindrical Pure Strain.**—Next suppose that the direction of the displacement of any point intersects perpendicularly a fixed right line.

In this case the strain is uniplanar, and we may write

$$u = x\psi, \quad v = y\psi, \quad w = 0. \tag{28}$$

Hence, for a pure strain, we have

$$\psi = \frac{1}{r}\frac{d\phi}{dr}, \tag{29}$$

where $$r^2 = x^2 + y^2.$$

Also

$$\left.\begin{aligned} a = \psi + \frac{x^2}{r}\frac{d\psi}{dr}, b = \psi + \frac{y^2}{r}\frac{d\psi}{dr}, \\ c = f = g = 0,\ h = \frac{xy}{r}\frac{d\psi}{dr} \end{aligned}\right\}. \tag{30}$$

The principal strains are plainly along r and perpendicular to r, and we have

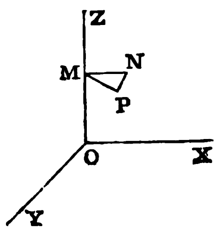
$$e_1 = \frac{d^2\phi}{dr^2},\ e_2 = \frac{1}{r}\frac{d\phi}{dr},\ e_3 = 0,\ \Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right). \tag{31}$$

This may be called a cylindrical strain.

28. **Pure Torsion.**—Suppose one end of a cylindrical body to be fixed, while its opposite end is twisted round the axis of the cylinder. Then the body is said to receive a pure torsion when any section, perpendicular to the axis, turns in its own plane through a small angle, whose amount varies directly as the distance of the plane of section from the fixed end of the body.

We can easily find expressions in this case for the displacements and the resulting strains at any point P.

Take the fixed axis for that of z, and the fixed end of the body for the plane of xy. From any point P draw PM perpendicular to OZ, and PN perpendicular to the plane of xz, then we have



$$OM = z, \quad PN = y, \quad MN = x.$$

Let $\Delta\theta$ denote the angle through which PM is made to twist, then, by hypothesis, $\Delta\theta = kz.$

Let l be the length of the cylindrical body, and let $\Delta\alpha$ be the angle of twist for the free end: then we have

$$\Delta\alpha = kl, \ \therefore \ \Delta\theta = \frac{z}{l}\Delta\alpha = \tau z, \quad \text{where } \tau = \frac{\Delta\alpha}{l}.$$

Accordingly, the displacements of P are given by the equations

$$u = -y\Delta\theta = -\tau yz, \quad v = x\Delta\theta = \tau xz, \quad w = 0. \qquad (32)$$

These equations represent what is called a *pure torsion*. The corresponding strains are

$$\left.\begin{array}{l} a = b = c = 0, \ 2h = \dfrac{du}{dy} + \dfrac{dv}{dx} = 0, \\ 2f = \dfrac{dv}{dz} = \tau x, \quad 2g = -\tau y \end{array}\right\}. \qquad (33)$$

EXAMPLES.

1. In the small strain of any continuous elastic body, investigate expressions for the diagonal elongations, facial distortions, and cubical dilatation of any elementary parallelepiped of its mass.

2. In a given strain find the pair of rectangular lines whose inclination is most altered by the strain.

3. Show that the shear corresponding to any two rectangular lines is equal to the difference between the elongations of the internal and external bisectors of the angles between the lines.

Let $$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = K,$$

represent the elongation quadric. If we transform to the internal and external bisectors of the axes between the axes of ξ and η, we have

$$\xi = \frac{1}{\sqrt{2}}(\xi' - \eta'), \quad \eta = \frac{1}{\sqrt{2}}(\xi' + \eta'),$$

and the part depending on ξ and η, transforms into

$$\tfrac{1}{2}a(\xi' - \eta')^2 + \tfrac{1}{2}b(\xi' + \eta')^2 + h(\xi'^2 - \eta'^2),$$

or

$$\tfrac{1}{2}(a + b + 2h)\xi'^2 + \tfrac{1}{2}(a + b - 2h)\eta'^2 - (a - b)\xi'\eta' + \&c. = K.$$

Hence we see that h is half the difference between the elongations in the directions of the axes of ξ' and η'.

4. In a given strain find the general condition that the angle between two rectangular lines, OP and OQ, should be unaltered by the strain.

They must be the axes of the corresponding section of the strain ellipsoid, &c.

SECTION III.—*Strain in Curvilinear Coordinates.*

29. **Orthogonal Curvilinear Coordinates.**—In some cases it is necessary to be able to express the constituents of a strain in polar or in some other system of coordinates which are not linear.

Thus, in general, if we suppose the equations

$$\phi_1(x, y, z) = \rho_1, \quad \phi_2(x, y, z) = \rho_2, \quad \phi_3(x, y, z) = \rho_3, \qquad (1)$$

where ρ_1, ρ_2, ρ_3 are arbitrary constants, to represent three systems of mutually orthogonal surfaces; then, for any point xyz, the variables ρ_1, ρ_2, ρ_3 may be called its *curvilinear coordinates*.

In this case the surfaces $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$, $\rho_3 = \text{const.}$, represent three orthogonal surfaces, which may be denoted by S_1, S_2, S_3, respectively.

For instance, in polar coordinates we have

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad \phi = \tan^{-1}\frac{y}{x}; \qquad (2)$$

then $r = \text{const.}$ represents a sphere, $\theta = \text{const.}$ represents a right cone, and $\phi = \text{const.}$ a plane. It is immediately seen that these systems of surfaces are mutually orthogonal.

Again, a system of confocal surfaces of the second degree intersect orthogonally; and so on in other cases.

Let us now suppose such an orthogonal system drawn through any point P; and let ds_1, ds_2, ds_3 be elements at P of the orthogonal system of curves in which they intersect. Then, since ρ_2 and ρ_3 are constant along the element ds_1, &c., we see that we must have

$$ds_1 = k_1 d\rho_1, \quad ds_2 = k_2 d\rho_2, \quad ds_3 = k_3 d\rho_3, \qquad (3)$$

where k_1, k_2, k_3 are functions of ρ_1, ρ_2, ρ_3.

It should be observed that the element ds_1 is normal to the surface S_1.

Again, let $l_1m_1n_1$ be the direction cosines of the tangent at P to ds_1, relative to the coordinate axes OX, OY, OZ; also, let $l_2m_2n_2$ be the corresponding cosines for ds_2, and $l_3m_3n_3$ those for ds_3.

Then, since the curves cut orthogonally, we have the system of equations

$$\left.\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 = 0 \\ l_1l_3 + m_1m_3 + n_1n_3 = 0 \\ l_2l_3 + m_2m_3 + n_2n_3 = 0 \end{aligned}\right\}; \tag{4}$$

and also

$$l_1m_2 - m_1l_2 = n_3, \quad l_3m_1 - m_3l_1 = n_2, \quad l_2m_3 - l_3m_2 = n_1, \tag{5}$$

with similar equations, got by interchange of suffixes.

30. **Equations of Transformation.**—Next, let ψ be any function of the coordinates, and we have

$$\left.\begin{aligned} \frac{d\psi}{dx} &= \frac{ds_1}{dx}\frac{d\psi}{ds_1} + \frac{ds_2}{dx}\frac{d\psi}{ds_2} + \frac{ds_3}{dx}\frac{d\psi}{ds_3} \\ &= l_1\,\frac{d\psi}{ds_1} + l_2\,\frac{d\psi}{ds_2} + l_3\,\frac{d\psi}{ds_3} \\ \text{and} \qquad \frac{d\psi}{dy} &= m_1\,\frac{d\psi}{ds_1} + m_2\,\frac{d\psi}{ds_2} + m_3\,\frac{d\psi}{ds_3} \\ \frac{d\psi}{dz} &= n_1\,\frac{d\psi}{ds_1} + n_2\,\frac{d\psi}{ds_2} + n_3\,\frac{d\psi}{ds_3} \end{aligned}\right\}. \tag{6}$$

Also

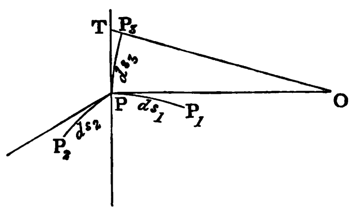
$$\left.\begin{aligned} \frac{1}{k_1}\frac{d\psi}{d\rho_1} = \frac{d\psi}{ds_1} = l_1\,\frac{d\psi}{dx} + m_1\,\frac{d\psi}{dy} + n_1\,\frac{d\psi}{dz} \\ \frac{1}{k_2}\frac{d\psi}{d\rho_2} = \frac{d\psi}{ds_2} = l_2\,\frac{d\psi}{dx} + m_2\,\frac{d\psi}{dy} + n_2\,\frac{d\psi}{dz} \\ \frac{1}{k_3}\frac{d\psi}{d\rho_3} = \frac{d\psi}{ds_3} = l_3\,\frac{d\psi}{dx} + m_3\,\frac{d\psi}{dy} + n_3\,\frac{d\psi}{dz} \end{aligned}\right\}. \tag{7}$$

Again, if by means of (1) we regard x, y, z as functions of ρ_1, ρ_2 and ρ_3, we have

$$\frac{dx}{d\rho_1} = k_1 \frac{dx}{ds_1} = l_1 k_1, \quad \frac{dx}{d\rho_2} = l_2 k_2, \quad \frac{dx}{d\rho_3} = l_3 k_3. \qquad (8)$$

31. **Theorem of Dupin.**—The following remarkable theorem was established by Dupin: *If three systems of surfaces be such that every surface of one system is cut orthogonally by all the surfaces of the two other systems, then the intersection of any two surfaces of different systems is a line of curvature on each* (Dupin, *Developments de Géométrie*, 1813, p. 330. See also Salmon's *Geometry of Three Dimensions*, p. 270.) From this theorem we see that ds_1 is an element of a line of curvature on each of the surfaces S_2 and S_3; in like manner ds_2 is an element of a line of curvature on the surfaces S_1 and S_3, &c.

Hence the normal at P_3 to the surface S_1 intersects PO the tangent to PP_1.



Again, the direction cosines of P_3O are

$$l_1 + \frac{dl_1}{ds_3} ds_3, \quad m_1 + \frac{dm_1}{ds_3} ds_3, \quad n_1 + \frac{dn_1}{ds_3} ds_3;$$

but, since PO and P_3O intersect, P_3O is perpendicular to the line l_2, m_2, n_2; and accordingly, by (4), we have

$$l_2 \frac{dl_1}{ds_3} + m_2 \frac{dm_1}{ds_3} + n_2 \frac{dn_1}{ds_3} = 0. \qquad (9)$$

And a system of similar equations can be found by interchanging *suffixes*.

32. **Expressions for Radii of Curvature.**—Again the angle between P_3O and the tangent PT is equal to $\frac{\pi}{2} + P_3OP$, and accordingly

$$l_3\left(l_1 + \frac{dl_1}{ds_3} ds_3\right) + m_3 \left(m_1 + \frac{dm_1}{ds_3} ds_3\right) + n_3 \left(n_1 + \frac{dn_1}{ds_1} ds_3\right)$$
$$= - \sin P_3OP = - \frac{ds_3}{OP};$$

hence

$$l_3 \frac{dl_1}{ds_3} + m_3 \frac{dm_1}{ds_3} + n_3 \frac{dn_1}{ds_3} = - \frac{1}{OP}.$$

Accordingly, we get, by aid of (4),

$$l_1 \frac{dl_3}{ds_3} + m_1 \frac{dm_3}{ds_3} + n_1 \frac{dn_3}{ds_3} = \frac{1}{OP}.$$

Let us now represent the principal radii of curvature at P for the surface S_1 by the notation ${}_2R_1$, ${}_3R_1$, where ${}_2R_1$ corresponds to the element ds_2, and ${}_3R_1$ to ds_3; and we have

$$l_1 \frac{dl_3}{ds_3} + m_1 \frac{dm_3}{ds_3} + n_1 \frac{dn_3}{ds_3} = \frac{1}{{}_3R_1}, \tag{10}$$

with corresponding expressions for the other principal radii of curvature at P.

Again, since

$$\frac{d^2x}{d\rho_1 d\rho_2} = \frac{d^2x}{d\rho_2 d\rho_1},$$

we get, from (8),

$$\frac{d}{d\rho_1} (l_2k_2) = \frac{d}{d\rho_2} (l_1k_1);$$

hence

$$\left.\begin{aligned} k_1 \frac{d}{ds_1} (l_2k_2) &= k_2 \frac{d}{ds_2} (l_1k_1) \\ k_1 \frac{d}{ds_1} (m_2k_2) &= k_2 \frac{d}{ds_2} (m_1k_1) \\ k_1 \frac{d}{ds_1} (n_2k_2) &= k_2 \frac{d}{ds_2} (n_1k_1) \end{aligned}\right\}. \tag{11}$$

If we multiply the first of these equations by l_1, the second by m_1, and the third by n_1, we readily get, by addition,

$$k_1\left(l_1\frac{dl_2}{ds_1}+m_1\frac{dm_2}{ds_1}+n_1\frac{dn_2}{ds_1}\right)=\frac{dk_1}{ds_2};$$

hence, from (10) we have

$$\frac{1}{{}_1R_2}=l_2\frac{dl_1}{ds_1}+m_2\frac{dm_1}{ds_1}+n_2\frac{dn_1}{ds_1}=-\frac{1}{k_1}\frac{dk_1}{ds_2}, \qquad (12)$$

along with similar equations got by change of suffixes.

33. **Expressions for Displacements.**—We shall now proceed to express the strains for any body in curvilinear coordinates.

We represent, as before, by u, v, w, the small displacements of P relative to the coordinate axes OX, OY, OZ: also by u_1, u_2, u_3, the corresponding component displacements of P relative to the directions of the tangents at P to ds_1, ds_2, ds_3, respectively; then we have the following equations

$$\left.\begin{aligned} u &= l_1u_1+l_2u_2+l_3u_3\\ v &= m_1u_1+m_2u_2+m_3u_3\\ w &= n_1u_1+n_2u_2+n_3u_2\end{aligned}\right\}, \qquad (13)$$

and also

$$\left.\begin{aligned} u_1 &= l_1u+m_1v+n_1w\\ u_2 &= l_2u+m_2v+n_2w\\ u_3 &= l_3u+m_3v+n_3w\end{aligned}\right\}. \qquad (14)$$

Again, if ρ_1, ρ_2, ρ_3 be the curvilinear coordinates of P before the strain, and ρ_1+a, ρ_2+a_2, ρ_3+a_3 its coordinates after the strain, then, by (3), we readily get

$$u_1=k_1a_1, \quad u_2=k_2a_2, \quad u_3=k_3a_3. \qquad (15)$$

These equations enable us to express the relations between the strains in terms of the displacements, $a_1a_2a_3$.

34. **Transformation of Normal Strains.**—Let, as before, $abcfgh$ be the constituents of the strain at P relative to

the axes of x, y, z. Also let $a_1 b_1 c_1 f_1 g_1 h_1$ be the corresponding constituents relative to the tangents to ds_1, ds_2, ds_3, respectively; then, by (15) Art. 10, we have

$$a_1 = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1$$

$$= l_1\left(l_1\frac{du}{dx} + m_1\frac{du}{dy} + n_1\frac{du}{dz}\right) + m_1\left(l_1\frac{dv}{dx} + m_1\frac{dv}{dy} + n_1\frac{dv}{dz}\right)$$

$$+ n_1\left(l_1\frac{dw}{dx} + m_1\frac{dw}{dy} + n_1\frac{dw}{dz}\right) = l_1\frac{du}{ds_1} + m_1\frac{dv}{ds_1} + n_1\frac{dw}{ds_1}, \text{ by (7)},$$

but

$$\frac{l_1 du}{ds_1} = \frac{d(ul_1)}{ds_1} - u\frac{dl_1}{ds_1}, \text{ \&c.};$$

hence by (14),

$$a_1 = \frac{du_1}{ds_1} - \left(u\frac{dl_1}{ds_1} + v\frac{dm_1}{ds_1} + w\frac{dn_1}{ds_1}\right),$$

also, by (13) and (9),

$$u\frac{dl_1}{ds_1} + v\frac{dm_1}{ds_1} + w\frac{dn_1}{ds_1}$$

$$= u_2\left(l_2\frac{dl_1}{ds_1} + m_2\frac{dm_1}{ds_1} + n_2\frac{dn_1}{ds_1}\right)$$

$$+ u_3\left(l_3\frac{dl_1}{ds_1} + m_3\frac{dm_1}{ds_1} + n_3\frac{dn_1}{ds_1}\right)$$

$$= \frac{u_2}{{}_1R_2} + \frac{u_3}{{}_1R_3}.$$

Hence we get

$$\left.\begin{aligned} a_1 &= \frac{du_1}{ds_1} - \frac{u_2}{{}_1R_2} - \frac{u_3}{{}_1R_3} \\ b_1 &= \frac{du_2}{ds_2} - \frac{u_3}{{}_2R_3} - \frac{u_1}{{}_2R_1} \\ c_1 &= \frac{du_3}{ds_3} - \frac{u_1}{{}_3R_1} - \frac{u_2}{{}_3R_2} \end{aligned}\right\}. \qquad (16)$$

35. Shearing Strains.

Again, by transformation of coordinates in the equation of the elongation quadric (Art. 10), we readily get

$$2h_1 = l_1\left(l_2\frac{du}{dx} + m_2\frac{du}{dy} + n_2\frac{du}{dz}\right) + l_2\left(l_1\frac{du}{dx} + m_1\frac{du}{dy} + n_1\frac{du}{dz}\right)$$

$$+ m_1\left(l_2\frac{dv}{dx} + m_2\frac{dv}{dy} + n_2\frac{dv}{dz}\right) + m_2\left(l_1\frac{dv}{dx} + m_1\frac{dv}{dy} + n_1\frac{dv}{dz}\right)$$

$$+ n_1\left(l_2\frac{dw}{dx} + m_2\frac{dw}{dy} + n_2\frac{dw}{dz}\right) + n_2\left(l_1\frac{dw}{dx} + m_1\frac{dw}{dy} + n_1\frac{dw}{dz}\right)$$

$$= l_1\frac{du}{ds_2} + l_2\frac{du}{ds_1} + m_1\frac{dv}{ds_2} + m_2\frac{dv}{ds_1} + n_1\frac{dw}{ds_2} + n_2\frac{dw}{ds_1}$$

$$= \frac{du_1}{ds_2} + \frac{du_2}{ds_1} - u\left(\frac{dl_1}{ds_2} + \frac{dl_2}{ds_1}\right) - v\left(\frac{dm_1}{ds_2} + \frac{dm_2}{ds_1}\right) - w\left(\frac{dn_1}{ds_2} + \frac{dn_2}{ds_1}\right).$$

Hence, by aid of (9), (10), and (13), we get immediately

$$\left.\begin{aligned} 2h_1 &= \frac{du_1}{ds_2} + \frac{du_2}{ds_1} + \frac{u_1}{{}_1R_2} + \frac{u_2}{{}_2R_1} \\ 2g_1 &= \frac{du_3}{ds_1} + \frac{du_1}{ds_3} + \frac{u_3}{{}_3R_1} + \frac{u_1}{{}_1R_3} \\ 2f_1 &= \frac{du_2}{ds_3} + \frac{du_3}{ds_2} + \frac{u_2}{{}_2R_3} + \frac{u_3}{{}_3R_2} \end{aligned}\right\}. \qquad (17)$$

These shears when transformed by (12), lead at once to the following :—

$$\left.\begin{aligned} 2f_1 &= k_2\frac{d}{ds_3}\left(\frac{u_2}{k_2}\right) + k_3\frac{d}{ds_2}\left(\frac{u_3}{k_3}\right) \\ 2g_1 &= k_3\frac{d}{ds_1}\left(\frac{u_3}{k_3}\right) + k_1\frac{d}{ds_3}\left(\frac{u_1}{k_1}\right) \\ 2h_1 &= k_1\frac{d}{ds_2}\left(\frac{u_1}{k_1}\right) + k_2\frac{d}{ds_1}\left(\frac{u_2}{k_2}\right) \end{aligned}\right\}.$$

By equations (3) and (15), these may be written in the ollowing simple form :—

$$\left.\begin{aligned} 2f_1 &= \frac{k_2}{k_3}\frac{da_2}{d\rho_3} + \frac{k_3}{k_2}\frac{da_3}{d\rho_2} \\ 2g_1 &= \frac{k_3}{k_1}\frac{da_3}{d\rho_1} + \frac{k_1}{k_3}\frac{da_1}{d\rho_3} \\ 2h_1 &= \frac{k_1}{k_2}\frac{da_1}{d\rho_2} + \frac{k_2}{k_1}\frac{da_2}{d\rho_1} \end{aligned}\right\}. \qquad (18)$$

36. **Case of Uniplanar Strain.**—In this case the lisplacements are all parallel to a fixed plane, which may be aken as that of xy. Also, we may take $\rho_3 = z$, and

$$\rho_1 = \phi_1(x, y), \quad \rho_2 = \phi_2(x, y),$$

vhere ϕ_1 and ϕ_2 represent two orthogonal systems of cylinlrical surfaces. Again, since the displacements are parallel o the plane of xy, we have $u_3 = 0$

Also,

$$\frac{1}{{}_1R_3} = 0, \quad \frac{1}{{}_2R_3} = 0, \quad \frac{1}{{}_3R_2} = 0, \quad \frac{1}{{}_3R_1} = 0.$$

Hence, if R_1 and R_2 be the radii of curvature at P of the urves ϕ_1 and ϕ_2, we have $c_1 = f_1 = g_1 = 0$, and

$$a_1 = \frac{du_1}{ds_1} - \frac{u_2}{R_2}, \; b_1 = \frac{du_2}{ds_2} - \frac{u_1}{R_1}, \; 2h_1 = \frac{du_1}{ds_2} + \frac{du_2}{ds_1} + \frac{u_1}{R_2} + \frac{u_2}{R_1}. \qquad (19)$$

37. **Strain in Polar Coordinates.**—In this case we ave

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Also

$$ds_1 = dr, \quad ds_2 = rd\theta, \quad ds_3 = r\sin\theta d\phi,$$

ccordingly

$$k_1 = 1, \quad k_2 = r, \quad k_3 = r\sin\theta.$$

Again, we see immediately that

$$\frac{1}{{}_1R_2} = \frac{1}{{}_1R_3} = \frac{1}{{}_2R_3} = 0, \quad \frac{1}{{}_2R_1} = \frac{1}{{}_3R_1} = -\frac{1}{r}, \quad \frac{1}{{}_3R_3} = -\frac{\cot\theta}{r}.$$

Let us now suppose that the coordinates r, θ, ϕ become after strain $r + a_1$, $\theta + a_2$, $\phi + a_3$, then we have

$$u_1 = a_1, \quad u_2 = ra_2, \quad u_3 = ra_3 \sin\theta, \tag{20}$$

and the constituents of the strain become

$$\left.\begin{aligned} &a_1 = \frac{da_1}{dr}, \quad b_1 = \frac{da_2}{d\theta} + \frac{a_1}{r}, \quad c_1 = \frac{da_3}{d\phi} + \frac{a_1}{r} + a_2 \cot\theta \\ &2h_1 = \frac{1}{r}\frac{da_1}{d\theta} + r\frac{da_2}{dr}, \quad 2g_1 = r\sin\theta\frac{da_3}{dr} + \frac{1}{r\sin\theta}\frac{da_1}{d\phi} \\ &2f_1 = \frac{1}{\sin\theta}\frac{da_2}{d\phi} + \sin\theta\frac{da_3}{d\theta} \end{aligned}\right\}. \tag{21}$$

38. **Strain in Cylindrical Coordinates.**—In this case the coordinates are r, θ, and z, and

$$x = r\cos\theta, \quad y = r\sin\theta, \quad ds_1 = dr, \quad ds_2 = rd\theta, \quad ds_3 = dz.$$

Hence we immediately have

$$\left.\begin{aligned} &a_1 = \frac{da_1}{dr}, \quad b_1 = \frac{da_2}{d\theta} + \frac{a_1}{r}, \quad 2h_1 = \frac{1}{r}\frac{da_1}{d\theta} + r\frac{da_2}{dr}, \\ &c_1 = \frac{dw}{dz}, \quad 2f_1 = \frac{1}{r}\frac{dw}{d\theta} + r\frac{da_2}{dz}, \quad 2g_1 = \frac{da_1}{dz} + \frac{dw}{dr}, \end{aligned}\right\}. \tag{22}$$

39. **Lemma.**—The following relation will be found of much utility in transformations from ordinary to curvilinear coordinates, viz.

$$m_1\frac{d\psi}{dz} - n_1\frac{d\psi}{dy} = l_3\frac{d\psi}{ds_2} - l_2\frac{d\psi}{ds_3},$$

where ψ is any function of xyz.

By (6) we have

$$m_1\frac{d\psi}{dz} - n_1\frac{d\psi}{dy}$$

$$= m_1\left(n_1\frac{d\psi}{ds_1} + n_2\frac{d\psi}{ds_2} + n_3\frac{d\psi}{ds_3}\right) - n_1\left(m_1\frac{d\psi}{ds_1} + m_2\frac{d\psi}{ds_2} + m_3\frac{d\psi}{ds_3}\right)$$

$$= l_3\frac{d\psi}{ds_2} - l_2\frac{d\psi}{ds_3}. \tag{23}$$

Similarly

$$n_1\frac{d\psi}{dx}-l_1\frac{d\psi}{dz}=m_3\frac{d\psi}{ds_2}-m_2\frac{d\psi}{ds_3};\ l_1\frac{d\psi}{dy}-m_1\frac{d\psi}{dx}=n_3\frac{d\psi}{ds_2}-n_2\frac{d\psi}{ds_3}.$$

40. **Vortex Displacements.**—If the angular displace-
ients of the strain at P, relative to the directions ds_1, ds_2,
s_3, be denoted by Θ_1, Θ_2, Θ_3, respectively, we have

$$\left.\begin{aligned}\Theta_1&=l_1\,\theta_1+m_1\,\theta_2+n_1\,\theta_3\\ \Theta_2&=l_2\,\theta_1+m_2\,\theta_2+n_2\,\theta_3\\ \Theta_3&=l_3\,\theta_1+m_3\,\theta_2+n_3\,\theta_3\end{aligned}\right\},\tag{24}$$

nd also

$$\left.\begin{aligned}\theta_1&=l_1\,\Theta_1+l_2\,\Theta_2+l_3\,\Theta_3\\ \theta_2&=m_1\,\Theta_1+m_2\,\Theta_2+m_3\Theta_3\\ \theta_3&=n_1\,\Theta_1+n_2\,\Theta_2+n_3\Theta_3\end{aligned}\right\}.\tag{25}$$

Accordingly, by (23), we have

$$2\Theta_1=l_1\frac{dw}{dy}-m_1\frac{dw}{dx}+n_1\frac{dv}{dx}-l_1\frac{dv}{dz}+m_1\frac{du}{dz}-n_1\frac{du}{dy}$$

$$=l_3\frac{du}{ds_2}+m_3\frac{dv}{ds_2}+n_3\frac{dw}{ds_2}-l_2\frac{du}{ds_3}-m_2\frac{dv}{ds_3}-n_2\frac{dw}{ds_3}.$$

.gain,

$$l_3\frac{du}{ds_2}=\frac{d}{ds_2}(ul_3)-u\frac{dl_3}{ds_2},\ \&c.$$

[ence, by (14),

$$\vartheta_1=\frac{du_3}{ds_2}-\frac{du_2}{ds_3}+u\left(\frac{dl_2}{ds_3}-\frac{dl_3}{ds_2}\right)+v\left(\frac{dm_2}{ds_3}-\frac{dm_3}{ds_2}\right)+w\left(\frac{dn_2}{ds_3}-\frac{dn_3}{ds_2}\right).$$

: we substitute from (13) for u, v, w, in terms of u_1, u_2, u_3,
e immediately get, by (10),

$$2\Theta_1=\frac{du_3}{ds_2}-\frac{du_2}{ds_3}+\frac{u_2}{{}_2R_3}-\frac{u_3}{{}_3R_2}.$$

By (12), this may be written

$$2\Theta_1 = \frac{1}{k_3}\frac{d(u_3k_3)}{ds_2} - \frac{1}{k_2}\frac{d(u_2k_2)}{ds_3} = \frac{1}{k_2k_3}\left(\frac{d(u_3k_3)}{d\rho_2} - \frac{d(u_2k_2)}{d\rho_1}\right). \tag{26}$$

In like manner, we get

$$2\Theta_2 = \frac{1}{k_1k_3}\left(\frac{d(u_1k_1)}{d\rho_3} - \frac{d(u_3k_3)}{d\rho_1}\right),$$

$$2\Theta_3 = \frac{1}{k_1k_2}\left(\frac{d(u_2k_2)}{d\rho_1} - \frac{d(u_1k_1)}{d\rho_2}\right).$$

Hence we see that the conditions in curvilinear coordinates for a pure strain are

$$\frac{d(u_1k_1)}{d\rho_3} = \frac{d(u_3k_3)}{d\rho_1}, \quad \frac{d(u_2k_2)}{d\rho_1} = \frac{d(u_1k_1)}{d\rho_2}, \quad \frac{d(u_3k_3)}{d\rho_2} = \frac{d(u_2k_2)}{d\rho_3}. \tag{27}$$

This result might have been anticipated since it expresses the condition that

$$u_1k_1d\rho_1 + u_2k_2d\rho_2 + u_3k_3d\rho_3$$

should be an exact differential; and it is easily seen that this follows from the property that $udx + vdy + wdz$ is an exact differential in the case of a pure strain.

Examples.

1. In any strain if $x'y'z'$, the coordinates of the strained position of a point xyz, be given by the equations

$$x' = x + \frac{dF}{dx}, \quad y' = y + \frac{dF}{dy}, \quad z' = z + \frac{dF}{dz},$$

where F is a function of xyz, show that the strain is a pure strain.

2. In the same case prove that we must have

$$x = x' + \frac{dF'}{dx'}, \quad y = y' + \frac{dF'}{dy'}, \quad z = z' + \frac{dF'}{dz'},$$

where F' is a function of $x'y'z'$.

Show also that F and F' are connected by the equation

$$F + F' = -\tfrac{1}{2}D^2 + \text{const.},$$

where

$$D = \frac{dF^2}{dx^2} + \frac{dF^2}{dy^2} + \frac{dF^2}{dz^2} = \frac{dF'^2}{dx'^2} + \frac{dF'^2}{dy'^2} + \frac{dF'^2}{dz'^2}.$$

3. In curvilinear coordinates prove that

$$udx + vdy + wdz = u_1\, ds_1 + u_2\, ds_2 + u_3\, ds_3.$$

4. Without assuming the equations of Art. 34, show directly that the condensation Δ is equal to

$$\frac{du_1}{ds_1} + \frac{du_2}{ds_2} + \frac{du_3}{ds_3} - u_1\left(\frac{1}{{}_2R_1} + \frac{1}{{}_3R_1}\right) - u_2\left(\frac{1}{{}_1R_2} + \frac{1}{{}_3R_2}\right) - u_3\left(\frac{1}{{}_1R_3} + \frac{1}{{}_2R_3}\right).$$

5. Prove that

$$\frac{d^2\rho_1}{dx^2} + \frac{d^2\rho_1}{dy^2} + \frac{d^2\rho_1}{dz^2} = \frac{1}{k_2k_3}\frac{d}{ds_1}\left(\frac{k_2k_3}{k_1}\right).$$

6. More generally show that, if V be any function,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \frac{d^2V}{ds_1^2} + \frac{d^2V}{ds_2^2} + \frac{d^2V}{ds_3^2} - \frac{dV}{ds_1}\left(\frac{1}{{}_2R_1} + \frac{1}{{}_3R_1}\right)$$
$$- \frac{dV}{ds_2}\left(\frac{1}{{}_1R_2} + \frac{1}{{}_3R_2}\right) - \frac{dV}{ds_3}\left(\frac{1}{{}_1R_3} + \frac{1}{{}_2R_3}\right).$$

7. This admits of being written also in the following form :—

$$\nabla^2 V = \frac{1}{k_2k_3}\frac{d}{ds_1}\left(k_2k_3\frac{dV}{ds_1}\right) + \frac{1}{k_3k_1}\frac{d}{ds_2}\left(k_3k_1\frac{dV}{ds_2}\right) + \frac{1}{k_1k_2}\frac{d}{ds_3}\left(k_1k_2\frac{dV}{ds_3}\right).$$

8. Give a direct proof of the last equation.

9. In the case of polar coordinates, prove that

$$\nabla^2 V \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\phi^2}.$$

10. If Δ be the cubical dilatation, prove that

$$\Delta = \frac{1}{k_1k_2k_3}\left\{\frac{d\,(k_1k_2k_3\alpha_1)}{d\rho_1} + \frac{d\,(k_1k_2k_3\alpha_2)}{d\rho_2} + \frac{d(k_1k_2k_3\alpha_3)}{d\rho_3}\right\}.$$

11. In a pure uniplanar strain if the equipotential curves be represented by the equation $r^n \cos n\theta = \text{constant}$, show that the displacement lines are represented by $r^n \sin n\theta = \text{const.}$

12. *OP*, *OQ*, *OR* are three rectangular lines drawn in an elastic body. If we suppose the body to receive a small strain, and that, while *OP* is fixed, the directions of *OQ* and *OR* are made to vary, prove that the sum of the squares of the shears of the angles *POQ* and *POR* is constant.

13. Hence show that the shear for the angle *POQ* varies as the cosine of the angle that the plane *POQ* makes with a fixed plane.

CHAPTER II.

STRESS.

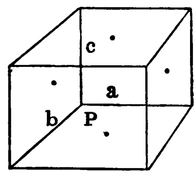
SECTION I.—*Homogeneous Stress.*

41. **Stress.**—We now proceed to consider the forces of elasticity which are brought into play when a body in equilibrium is in a given condition of strain. In such case any portion of the body may be regarded as in equilibrium under the influence of the internal and external forces which act on it. The forces over its surface which any portion of the body experiences from the action of its surrounding matter are called the superficial stresses on that portion. Also, since the equilibrium of any portion of a body is unaltered by conceiving that portion as being perfectly rigid, the general equations of equilibrium for a rigid body apply to every portion of an elastic body in a given state of strain, when all the forces which act on it are taken into account.

The student should observe that in all our applications, unless the contrary is stated, we assume that the bodies continue perfectly elastic (see Art. 1) within the limits of stress and strain involved in the investigations.

42. **Homogeneous Stress.**—The stress on any plane section is homogeneous when it is uniformly distributed over the section; in this case each element of the area bears a proportional part of the whole stress. Accordingly a stress is said to be homogeneous when equal and parallel portions of the body experience equal and parallel stresses on all equal and parallel portions of area. Also, since in this case the stresses at each point of any plane section of the body are parallel, we see that the stresses on any plane area in the interior of a body have for their resultant a single force, applied at the centroid of that area. In all cases the stress on any plane area is estimated by its amount per *unit of area.*

43. **Conditions of Equilibrium.**—Suppose any rectangular parallelepiped taken in the interior of an elastic body, with its edges parallel to a system of rectangular axes; and let a be the length of its edge parallel to the axis of x, b that parallel to y, and c to z.



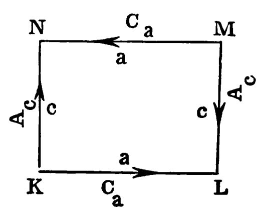
Now, for a homogeneous stress, in considering the equilibrium of the parallelepiped, we may suppose the entire stress action on any face as applied at the centre of that face.

It will be shown subsequently that for a homogeneous stress the external forces which act on the parallelepiped must be wholly superficial.

Accordingly we may regard the parallelepiped as in equilibrium under the action of three pairs of equal, opposite, and parallel forces, applied respectively at the middle points of its six faces.

Again, the stress on any face may be resolved into three components parallel respectively to the edges of the parallelepiped. Suppose these, for the face ab, when referred to unit of area to be denoted by C_a, C_b, C_c, respectively; and adopt a corresponding notation, $A_a, A_b, A_c, B_a, B_b, B_c$, for the stresses on the other faces. Of this system the stresses A_a, B_b, C_c are normal to the corresponding faces, while the others act in the planes of the faces, and are called *tangential* or *shearing* stresses.

44. **Relations between the Tangential Stresses.**—Let us now draw a rectangle passing through the middle points of two pairs of opposite faces. Let $KLMN$ be such a rectangle, parallel to the plane of xz; then, confining our attention to stresses parallel to that plane, we denote them per unit area by C_a, A_c as in the figure; omitting the normal components which are equal and directly opposed. Accordingly the pair of equal and opposite forces $C_a \times ab$, at the distance c



apart, must be in equilibrium with the pair $A_c \times bc$ at the distance a apart. Hence

$$C_a \times abc = A_c \times abc,$$

we consequently have

$$C_a = A_c, \text{ and also } B_c = C_b, \; A_b = B_a. \qquad (1)$$

Hence we infer that for any pair of mutually rectangular planes the stress action on either plane, in a direction perpendicular to the other, is the same for both planes.

We thus see that in homogeneous stress the entire stress action on the surface of a parallelepiped is equivalent to three pairs of equal and opposite normal forces, along with three pairs of tangential or shearing forces, in these faces respectively.

From analogy to the notation already adopted for a strain, we shall denote these stresses by the letters

$$A, B, C, F, G, H.$$

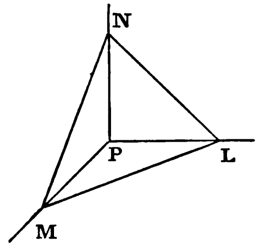
Of these A, B, C are normal stresses, and the others tangential.

For instance, $F = B_c = C_b$ represents, per unit of area, a shearing stress perpendicular to x, in the plane of xy; and also an equal shearing stress perpendicular to x, in the plane of xz. Similarly for the stresses G and H.

If the exterior normal stress A acts *outwards* on the face of the parallelepiped, it is called a *traction*; if inwards, it is called a *pressure*. It is evident that a pair of equal and opposite tractions tend to elongate the corresponding edge of the parallelepiped, and to produce a positive strain, or increase of strain. The stress A is accordingly *positive when it is a traction*, and *negative when a pressure*. A traction is sometimes called also a *pull*, and a pressure a *push*. To an external stress A always corresponds, as already stated, an equal and opposite *internal stress*, arising from the elasticity of the body.

45. **Stress on any Plane.**—Suppose any plane drawn, intersecting the edges of the parallelepiped in LMN, respec-

tively; and let us consider the equilibrium of the tetrahedron, under the action of the stresses on its four faces.



Let T denote the stress, per unit area, for the face LMN, and let $\lambda\mu\nu$ be the angles that the direction of T makes with the directions of the corresponding edges.

Also let $\alpha\beta\gamma$ be the direction angles of the perpendicular to the plane LMN.

Then resolving the stresses parallel to the edge PL, we have, for equilibrium,

$$T\Delta \cos \lambda = A\Delta \cos \alpha + H\Delta \cos \beta + G\Delta \cos \gamma, \quad (2)$$

where Δ represents the area of the triangle LMN.

From this and the corresponding equations we have

$$\left.\begin{aligned} T \cos \lambda &= A \cos \alpha + H \cos \beta + G \cos \gamma, \\ T \cos \mu &= H \cos \alpha + B \cos \beta + F \cos \gamma, \\ T \cos \nu &= G \cos \alpha + F \cos \beta + C \cos \gamma. \end{aligned}\right\} \quad (3)$$

These equations enable us to determine the stress T both in magnitude and in direction.

If we now suppose the plane to pass through P, these equations still hold good, and enable us to find the stress on any plane drawn through P.

Again, if we suppose each edge of the tetrahedron to become indefinitely small it is readily seen that equations (3) still hold good even when external forces are supposed to act on the volume of the tetrahedron.

For let Xdm, Ydm, Zdm represent the components of the external force which acts on dm, the elementary mass of the tetrahedron; and let $dm = \rho dV$, where dV denotes the volume of the tetrahedron.

Then instead of equation (2) we should have

$$T\Delta \cos \lambda = A\Delta \cos \alpha + H\Delta \cos \beta + G\Delta \cos \gamma + X\rho dV,$$

or

$$T \cos \lambda = A \cos \alpha + H \cos \beta + G \cos \gamma + X\rho \frac{dV}{\Delta}. \quad (4)$$

Now, let p be the distance of P from the plane LMN, then by the ordinary expression for the volume of a tetrahedron we have

$$dV = \tfrac{1}{3} p\Delta, \text{ or } \frac{dV}{\Delta} = \tfrac{1}{3} p.$$

Hence $\dfrac{dV}{\Delta}$ vanishes along with p; and we see that equations (3) still hold good when external forces act on the body, for pressures at the point P on any plane drawn through P.

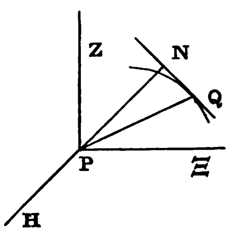
46. **Constituents of a Homogeneous Stress.**—From equations (3) we see that for a homogeneous stress the stress on any plane is completely determined when the direction of the plane is given, as also the stresses $ABCFGH$; accordingly, as in the corresponding case of strain, these six quantities may be called the six constituents of the stress.

47. **Stress Quadric.**—Again, as in the corresponding case of strain, equations (3) readily admit of geometrical interpretation.

For, let the quadric

$$U = A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta = K \quad (5)$$

be taken; then, to find the stress on any plane, let PQ, drawn normal to the plane, intersect in Q the quadric U, and let PN be perpendicular to the tangent plane to U, drawn at the point Q, whose coordinates are ξ, η, ζ; then, by equations (3) we get, as in Art. 16,



$$\cos\lambda : \cos\mu : \cos\nu = \frac{dU}{d\xi} : \frac{dU}{d\eta} : \frac{dU}{d\zeta}. \quad (6)$$

Hence we see, as in Art. 16, that PN is the direction of the resultant stress T.

Again, let $PQ = r$, $PN = p$, and we get, from (3),

$$Tr \cos\lambda = A\xi + H\eta + G\zeta,$$

$$Tr \cos\mu = H\xi + B\eta + F\zeta,$$

$$Tr \cos\nu = G\xi + F\eta + C\zeta.$$

Hence $$Tr\,(\xi \cos\lambda + \eta \cos\mu + \zeta \cos\nu) = K,$$

r $$T = \frac{K}{pr}. \tag{7}$$

Again, if N be the stress normal to the plane, we have

$$N = T\frac{p}{r} = \frac{K}{r^2}. \tag{8}$$

This may be written

$$N = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma + 2F\cos\beta\cos\gamma$$
$$+ 2G\cos\gamma\cos\alpha + 2H\cos\alpha\cos\beta. \tag{9}$$

Again, if $N = 0$ the corresponding normals lie on the symptote cone to the stress quadric U.

From the foregoing property this cone (when it exists) is alled the cone of shearing stress; and the stress action on a lane, perpendicular to any edge of this cone, reduces to a hearing stress solely. In order that this cone should exist he stress quadric must be an hyperboloid.

Again, the lines for which the normal stress is constant E) lie on the cone

$$(A-E)\,\xi^2 + (B-E)\,\eta^2 + (C-E)\,\zeta^2 + 2F\eta\zeta$$
$$+ 2G\zeta\xi + 2H\xi\eta = 0. \tag{10}$$

Compare Arts. 10 and 16.)

48. **Principal Stresses.**—In order that the stress T hould be normal to the plane on which it acts, the line PQ ust be normal to the stress quadric, *i. e.* PQ must be a rincipal axis of U. Hence we see there are in general ree, and but three, planes at any point for which the whole ress is normal to the plane. These planes are called the rincipal planes of the stress, and the corresponding stresses re called its principal stresses.

To find the principal stresses, let $\lambda = \alpha$, $\mu = \beta$, $\nu = \gamma$ in quations (3), and we get by elimination

$$\begin{vmatrix} A-T, & H, & G, \\ H, & B-T, & F, \\ G, & F, & C-T, \end{vmatrix} = 0. \tag{11}$$

The roots of this equation represent accordingly the principal stresses.

The stress quadric when referred to its axes may be written

$$E_1\xi^2 + E_2\eta^2 + E_3\zeta^2 = K, \qquad (12)$$

in which E_1, E_2, E_3 represent the principal stresses.

In this case equations (3) assume the following simple form

$$T\cos\lambda = E_1\cos\alpha,\quad T\cos\mu = E_2\cos\beta,\quad T\cos\nu = E_3\cos\gamma. \quad (13)$$

49. **Invariants of the Stress.**—Again, as in Art. 18, the quantities

$$A+B+C,\ AB+AC+BC-F^2-G^2-H^2,$$

and the determinant

$$\Delta = \begin{vmatrix} A, & H, & G, \\ H, & B, & F, \\ G, & F, & C, \end{vmatrix}$$

are invariants of the stress.

50. **Uniplanar Stress.** — Whenever the forces and stresses are all parallel to one plane, we may take that plane as the plane of $\xi\eta$, and the equation of the stress quadric becomes

$$A\xi^2 + B\eta^2 + 2H\xi\eta = K. \qquad (14)$$

In uniplanar stress one root of (11) must be zero, and consequently the determinant $\Delta = 0$.

Examples.

1. Prove that a shearing stress on two rectangular planes produces equal normal stresses, of opposite signs, on two planes which bisect the angles between the given rectangular planes.

This follows from the property that the equation

$$2Hxy = K,$$

when transferred to its axes, becomes

$$H(x^2 - y^2) = K.$$

(See Art. 10.)

2. If A, B, H be the constituents of a uniplanar stress, find the corresponding principal stresses.

Here we have

$$E_1 + E_2 = A + B, \text{ and } E_1E_2 = AB - H^2, \text{ \&c.}$$

SECTION II.—*Heterogeneous Stress.*

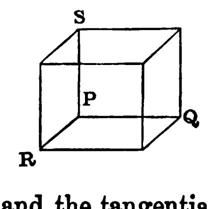
51. **Heterogeneous Stress.**—In the general case of elastic solids, subject to external force, the stress action on any plane element supposed taken in the interior of the body varies from point to point in the plane. We shall assume that such stresses are continuous quantities, *i.e.* we suppose that the differences between the components of stress on two equal small areas, taken indefinitely close to each other in a plane, are very small in comparison with the stresses themselves. Hence, if we neglect small quantities, we may suppose that the distribution of stress is constant over a very small plane area dS. Accordingly, as in the corresponding case of strain (Art. 20), we may treat the stresses as homogeneous throughout an indefinitely small volume. For instance, the whole normal stress on the elementary area $dy\,dz$ at a point P may, as in Art. 44, be denoted by $A\,dy\,dz$, where A represents as before the stress uniformly distributed over a unit of area.

We thus see that we may still denote the constituents of stress by the letters $ABCFGH$, provided we regard these quantities as varying from point to point in the solid.

Thus we readily see that equations (3) for the stress at P, on any plane element passing through P, still hold good. Also that the properties derived from the stress quadric at any point are applicable to heterogeneous stress.

Again, we can proceed from the properties of homogeneous to those of heterogeneous stress, in the same manner that in Chapter I. we proceeded from homogeneous to heterogeneous strain, provided we consider the stresses $ABCFGH$ as varying continuously from point to point, *i.e.* as functions of the coordinates of the point.

52. **Equations of Equilibrium.**—We next proceed to determine the general equations of equilibrium. At any point P, whose coordinates are xyz, suppose an indefinitely small, rectangular parallelepiped drawn, the lengths of whose edges are represented by Δx, Δy, and Δz, then by our previous notation the normal stress on



the face RPS is represented by $A\,\Delta y\,\Delta z$; and the tangential

stresses by $H\,\Delta y\,\Delta z$ and $G\Delta y\,\Delta z$; and similarly for the stresses for the faces RPQ and SPQ.

Again, since A is a function of xyz, the components of traction, per unit of area, on the face which is opposite to RPS are

$$A + \frac{dA}{dx}\,\Delta x, \quad H + \frac{dH}{dx}\,\Delta x, \quad G + \frac{dG}{dx}\,\Delta x,$$

since x is the only coordinate which is supposed to vary in this case. The components of stress on the other faces can be likewise expressed.

Hence, for the equilibrium of the elementary parallelepiped, there must be equilibrium between the stresses acting on the faces of the parallelepiped and the external forces acting on its element of mass. Accordingly the entire stress action parallel to the axis of x must equilibrate the external force, Xdm, parallel to that axis. Again, the equal and opposite forces, $A\,\Delta y\,\Delta z$, &c., equilibrate; and we see that the entire stress action parallel to the axis of x is represented by

$$-\left(\frac{dA}{dx} + \frac{dH}{dy} + \frac{dG}{dz}\right)\Delta x\,\Delta y\,\Delta z;$$

equating this to $X\,dm$, *i.e.* to $X\rho\,\Delta x\,\Delta y\,\Delta z$, and taking the corresponding equations for the other axes, we get

$$\left.\begin{aligned} \frac{dA}{dx} + \frac{dH}{dy} + \frac{dG}{dz} + \rho X &= 0, \\ \frac{dH}{dx} + \frac{dB}{dy} + \frac{dF}{dz} + \rho Y &= 0, \\ \frac{dG}{dx} + \frac{dF}{dy} + \frac{dC}{dz} + \rho Z &= 0. \end{aligned}\right\} \qquad (1)$$

In the particular case of uniplanar stress, parallel to the plane of xy suppose, these reduce to

$$\left.\begin{aligned} \frac{dA}{dx} + \frac{dH}{dy} + \rho X &= 0, \\ \frac{dH}{dx} + \frac{dB}{dy} + \rho Y &= 0. \end{aligned}\right\} \qquad (2)$$

It should be observed that in the foregoing proof we have throughout neglected forces which are infinitely small in comparison with those retained.

53. **Relation between Stress and Strain.**—For an elastic substance in equilibrium, under a constant temperature, any state of strain at each point corresponds to a certain state of stress at the point. Also, when the strain at any point is given the corresponding stress at the point may be regarded as perfectly determinate for a given substance, depending on the nature of the substance, &c. We may consequently, for any substance, regard the stresses $ABCFGH$ as functions of the strains $abcfgh$ along with constants depending on the nature and properties of the substance. We shall show subsequently that for the small strains of a perfectly elastic substance the stresses may be taken as *linear* functions of the strains.

54. **Lines of Stress.**—In Art. 48 we have shown that at each point in an elastic body there are in general three directions for which the corresponding shearing stresses disappear, viz. the directions of the axes of the corresponding stress quadric. Suppose PP' to be an element taken on one of these principal axes at the point P; again let $P'P''$ be a corresponding element at P', &c., we thus see that we can draw a continuous curve $PP'P''P'''\ldots$ composed of such elements.

By this means we can conceive three systems of such curves drawn in space: these systems plainly cut everywhere orthogonally; and if we consider an elementary volume whose conterminous edges are respectively elements of this system of curves, the corresponding elementary volume is approximately a parallelepiped, and the stress over its surface consists solely of normal tractions across the corresponding faces of the elementary volume.

The curves here indicated are called *lines of stress.* The differential equations of a line of stress are readily seen, from Art. 45, to be

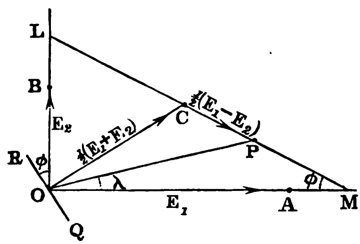
$$\frac{Adx + Hdy + Gdz}{dx} = \frac{Hdx + Bdy + Fdz}{dy} = \frac{Gdx + Fdy + Cdz}{dz}. \quad (3)$$

55. **Spheroidal and Uniplanar Stress, Rankine's Construction.**—If a plane passes through one of the principal axes of the stress quadric, the axis of E_3 suppose, equations (13), Art. 48, with a slight change of notation, may be written in the simple form

$$T\cos\lambda = E_1\cos\phi, \quad T\sin\lambda = E_2\sin\phi. \qquad (4)$$

These equations lead immediately to a remarkable construction for the magnitude and direction of the stress on any plane through a principal axis.

For, let OA and OB be taken to represent in magnitude and direction the stresses E_1 and E_2, respectively; and let



RQ be the trace on the plane AOB of any plane passing through the axis of E_3; then, to construct T the stress on this plane, draw OC perpendicular to RQ, and measure OC representing the force $\frac{1}{2}(E_1 + E_2)$; draw $CM = OC$, and produce it to meet OB in L, then $LM = 2OC = OA + OB$. If we take $LP = OA$, and join OP, then we can show that OP represents the stress T both in magnitude and direction.

For we have from the figure

$$OP\sin POM = PM\sin\phi, \quad OP\sin POL = PL\cos\phi. \qquad (5)$$

Hence, from (4), we see that $POM = \lambda$, and $OP = T$.

Accordingly, P is the resultant of the force $\frac{1}{2}(E_1 + E_2)$, normal to the plane, along with the force $\frac{1}{2}(E_1 - E_2)$ acting parallel to the line LM.

Again, if x, y be the coordinates of P, equations (5) give at once

$$\frac{x^2}{OA^2} + \frac{y^2}{OB^2} = 1. \qquad (6)$$

This shows that the point P lies on the ellipse whose semi-axes are E_1 and E_2. By aid of this ellipse the stress on any plane which passes through an axis of the stress quadric can be found when the principal stresses are known.

Again, when two of the principal stresses are equal, the preceding method enables us to determine the stress on any plane.

For, if we suppose $E_2 = E_3$, then any plane may be regarded as passing through a line of principal stress: for the line in which the plane intersects the plane of equal stress may be taken as the direction of an axis of the stress quadric.

Accordingly, the preceding results apply to the determination of the stress on any plane whenever the stress quadric is of revolution, such a stress may be styled a *spheroidal* stress.

Again, in uniplanar stress it is obvious that this method enables us to determine the stress on any plane. The foregoing construction is due to Rankine: see his "Applied Mechanics," pp. 101–112.

It is readily seen that the directions OP and OQ are conjugate to each other, *i. e.* that the stress on the perpendicular plane passing through OP is in the direction OQ.

This follows at once from the equation

$$\frac{\tan\lambda}{\tan\phi} = \frac{E_2}{E_1},$$

for if we take the corresponding equation for any other plane we have

$$\frac{\tan\lambda}{\tan\phi} = \frac{\tan\lambda'}{\tan\phi'}. \tag{7}$$

Hence, if

$$\phi' = \tfrac{1}{2}\pi + \lambda, \text{ we get } \lambda' = \tfrac{1}{2}\pi + \phi,$$

which proves the relation.

Such stresses have been styled *conjugate* stresses by Rankine. Also, if T and T' be the stresses corresponding to such a conjugate pair of planes, we immediately see that

$$TT' = E_1 E_2. \tag{8}$$

E 2

The ellipse whose axes are OA and OB has been styled the *ellipse of stress* by Rankine; and the whole theory of the distribution of spheroidal and of uniplanar stress reduces to the discussion of the properties of this ellipse.

Again, the angle COP is that between the direction of the resultant stress and the normal to the plane on which it acts. This angle is called the *obliquity* of the resulting stress when the principal stresses are *like*. This obliquity is plainly a maximum when the angle OPC is a right angle. Also, if γ denote its maximum value, we have

$$\sin\gamma = \frac{E_1 - E_2}{E_1 + E_2};$$

or

$$\frac{E_2}{E_1} = \frac{1 - \sin\gamma}{1 + \sin\gamma}. \qquad (9)$$

Again, we have in this case

$$T^2 = E_1 E_2. \qquad (10)$$

Hence T is a mean proportional between the principal stresses.

Also, in the same case we have $\sin\gamma = \cos 2\phi$.

Consequently,

$$\frac{E_2}{E_1} = \frac{1 - \cos 2\phi}{1 + \cos 2\phi} = \tan^2\phi.$$

This determines the position of the plane for which the resultant stress makes the greatest angle with the normal to the plane.

One important application of Rankine's construction is to the consideration of the stability of earthworks. Thus, from the elementary consideration of friction it appears immediately that the surface of a mass of loose earth under the action of gravity can only remain permanently at rest when the inclination to the horizon of the tangent plane at each point on the surface is less than, or equal to, the angle of friction for the loose earth. The angle of friction is accordingly called in this case the *angle of repose* of the earth. We shall denote this angle by ε.

Again, for the equilibrium of any particle in the interior of the loose earth it is evident that if the obliquity of the resultant stress on any plane drawn through the point is greater than the angle of repose, slipping will take place along the plane.

Consequently, in order that the earth should be undisturbed at any interior point, it is necessary that the maximum obliquity of the stress at the point should not be greater than the angle of friction, *i.e.* γ must not exceed ε. Hence, assuming the stress to be spheroidal, and that E_1 and E_2 denote the principal stresses for the loose earth at the point, we must have $\frac{E_2}{E_1}$ not less than $\frac{1 - \sin \varepsilon}{1 + \sin \varepsilon}$; or, in the limit,

$$\frac{E_2}{E_1} = \frac{1 - \sin \varepsilon}{1 + \sin \varepsilon}, \tag{11}$$

where ε is the angle of repose.

For earth whose upper surface is *horizontal* the vertical stress at any interior point cannot exceed W the weight of the earth, and the horizontal stress may be assumed to be the same all round at the same point. In this case if the horizontal stress be not less than $\frac{1 - \sin \varepsilon}{1 + \sin \varepsilon} W$, the loose earth cannot *spread*. On the other hand, if the horizontal stress be less than $\frac{1 + \sin \varepsilon}{1 - \sin \varepsilon} W$, the earth cannot *heave* up.

For example, to find the least depth, consistent with equilibrium, to which the foundation of a wall should be sunk in loose earth.

Let d be the depth of the foundation, h the height of the wall over the surface of the ground, and w the weight per unit volume of the wall, then we have $E_1 = w(h + d)$. Hence the limiting value of E_2, the horizontal stress, is given by

$$E_2 = E_1 \frac{1 - \sin \varepsilon}{1 + \sin \varepsilon} = w(h + d) \frac{1 - \sin \varepsilon}{1 + \sin \varepsilon}. \tag{12}$$

Again, for the earth adjacent to the wall, the vertical pressure at the depth d is represented by $w_1 d$, where w_1 is the weight of a unit volume of the earth; hence if E_1', E_2' be the principal stresses for the loose earth at the depth d, we have

$$E_1' = w_1 d.$$

Also, in order that the earth should not heave up, we have *in the limit*

$$E_2' = E_1' \frac{1 + \sin \varepsilon}{1 - \sin \varepsilon} = w_1 d \frac{1 + \sin \varepsilon}{1 - \sin \varepsilon}. \qquad (13)$$

Moreover, for the equilibrium of the loose earth at the depth d, we must have $E_2' = E_2$, and consequently

$$\frac{w (h + d)}{w_1 d} = \left(\frac{1 + \sin \varepsilon}{1 - \sin \varepsilon}\right)^2;$$

or

$$\sigma \left(\frac{h}{d} + 1\right) = \left(\frac{1 + \sin \varepsilon}{1 - \sin \varepsilon}\right)^2, \qquad (14)$$

where σ is the ratio of the specific gravities of the wall and of the loose earth.

The least depth of the wall can be readily calculated in any particular case from equation (14).

The student will find interesting examples of this method in Rankine's work already referred to, also in Alexander's "Elementary Applied Mechanics."

Examples.

1. If from any point O lines OP, &c., be drawn which represent in magnitude and direction the stresses for all planes drawn through O, find the locus of the point P.

Ans. The ellipsoid $\frac{\xi^2}{E_1^2} + \frac{\eta^2}{E_2^2} + \frac{\zeta^2}{E_3^2} = 1$.

2. If the stress on every elementary plane drawn through a point P is normal to the corresponding plane, show that the stress is constant for all planes at the point P.

A substance which possesses the property here stated is called a perfect liquid. If p be the elastic force at any point in a liquid, the general equations of equilibrium become

$$\frac{dp}{dx} + \rho X = 0, \quad \frac{dp}{dy} + \rho Y = 0, \quad \frac{dp}{dz} + \rho Z = 0.$$

The following examples can be readily solved from the fundamental property of the stress quadric, viz. that if its equation be transferred to any new rectangular system of axes, the coefficients of the terms in the transformed equation represent the stress system corresponding to the new axes.

3. Give the six constituents of stress at a point for one system of rectangular axes; find the corresponding constituents of stress relative to any other rectangular system.

4. Show that a shearing stress produces equal and opposite normal stresses on the planes which bisect the angles between those of the shearing stress.

5. If there be no normal stress on any plane which passes through a fixed line, show that the stress is equivalent to a normal stress on the plane perpendicular to the line, along with a shearing stress, on two rectangular planes.

6. In a uniplanar stress, find the pair of rectangular planes for which the shearing stress is a maximum.

7. In any given stress system find the planes passing through a given line, for which the shearing stress is a maximum.

8. In the same case find the planes for which the shearing stress vanishes.

9. In any stress find the pair of planes for which the shearing stress is a maximum, and its maximum value. *Ans.* $\frac{1}{2}(E_1 - E_3)$.

10. Show, by Green's theorem, that the six equations of equilibrium for a rigid body hold good in the case of any elastic solid.

Here

$$\iiint X\rho\, dxdydz = -\iiint\left(\frac{dA}{dx} + \frac{dH}{dy} + \frac{dG}{dz}\right) dxdydz.$$

But, by Green's theorem (*Integral Calculus*, Art. 226), the latter integral is equal to

$$\iint(lA + mH + nG)\, dS,$$

taken over the bounding surface or surfaces, where lmn are the direction cosines of the normal at dS to the bounding surface.

Again, by (3), Art. 45,

$$\iint(lA + mH + nG)\, dS = \iint T\cos\lambda dS.$$

Hence, we get

$$\iiint X\rho\, dxdydz + \iint T\cos\lambda dS = 0,$$

along with two similar equations.

Again,

$$\iiint (yZ - zY)\,\rho\, dxdydz$$

$$= -\iiint \left\{ y\left(\frac{dG}{dx} + \frac{dF}{dy} + \frac{dC}{dz}\right) - z\left(\frac{dH}{dx} + \frac{dB}{dy} + \frac{dF}{dz}\right) \right\} dxdydz$$

$$= -\iiint \left\{ \frac{d}{dx}(Gy - Hz) + \frac{d}{dy}(Fy - Bz) + \frac{d}{dz}(Cy - Fz) \right\} dxdydz$$

$$= -\iint \{ y\,(lG + mF + nC) - z\,(lH + mB + nC) \}\, dS$$

$= -\iint T\,(y \cos\mu - z \cos\nu)\, dS$, by Green's theorem.

The six general equations of equilibrium are thus proved.

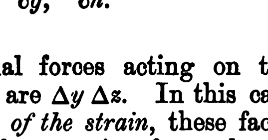
The above may be regarded as furnishing us with an independent proof of Green's theorem, based on the elementary principles of mechanics.

CHAPTER III.

CONNECTION BETWEEN STRESS AND STRAIN.

SECTION I.—*Work and Potential Energy.*

56. **Work done in a Small Change of Strain.**—
et us now suppose the state of stress of a perfectly elastic olid, *its temperature remaining constant*, to change coninuously. In this case the corresponding strain at each oint will also change continuously; but be such that at ach stage the strain is that which belongs to the correspondng stress at the instant. We proceed to investigate the vork done against the elastic forces of the body in producing uch a change in its state, the body being supposed perfectly lastic during the change. Let us irst consider the elementary work erformed by the stresses which ct on the faces of an elementary arallelepiped $\Delta x \Delta y \Delta z$, in prolucing the small change of strain epresented by



$$\delta a, \quad \delta b, \quad \delta c, \quad \delta f, \quad \delta g, \quad \delta h.$$

We shall first take the normal forces acting on the pposite pair of faces whose areas are $\Delta y \, \Delta z$. In this case *eglecting an indefinitely small part of the strain*, these faces nay be regarded as acted upon by a pair of equal and pposite forces, each equal to $A \, \Delta y \, \Delta z$. Accordingly (see *Dynamics*, Art. 127), the elementary work done by this tress is represented by

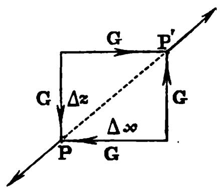
$$A \, \Delta y \, \Delta z \, \delta (\Delta x),$$

where $\delta(\Delta x)$ denotes the elongation of Δx arising from the strain; but, by hypothesis, $\delta a = \frac{\delta(\Delta x)}{\Delta x}$; hence the preceding element of work is represented by

$$A\,\delta a \times \Delta x\,\Delta y\,\Delta z. \tag{1}$$

The elements of work due to the stresses B and C are, in like manner, found to be $B\,\delta b \times \Delta x\,\Delta y\,\Delta z$, and $C\,\delta c \times \Delta x \Delta y \Delta z$.

We shall next find the element of work done by a shearing stress acting on two opposite pairs of faces of the parallelepiped. It is readily seen that on the faces $\Delta y\,\Delta z$, and $\Delta y\,\Delta x$, the tangential forces are $G\,\Delta z\,\Delta y$, and $G\,\Delta x\,\Delta y$, respectively. Now the resultant of these forces, regarded as acting along Δz and Δx, is a force acting in the line, PP', whose intensity is $G\,\Delta y \times PP'$. Similarly, the resultant of the parallel stresses at P' is an equal and opposite force acting along PP'. Hence the work done by this system of shearing stresses is represented by



$$G\,\Delta y\,PP'\,\delta(PP'),$$

where $\delta(PP')$ is the elongation of the diagonal PP', but, by Art. 12, the shear of the angle at P is represented by $2\delta g$; also we see immediately that

$$PP'.\,\delta(PP') = \Delta x\,\Delta z \times 2\delta g.$$

Hence, the work done by the shearing stress in causing the small shear δg is represented by

$$2G \times \delta g \times \Delta x\,\Delta y\,\Delta z.$$

Accordingly, by the principle of superposition, the total work done by all the stresses on the element of volume $\Delta x\,\Delta y\,\Delta z$ is represented by

$$(A\,\delta a + B\,\delta b + C\,\delta c + 2F\,\delta f + 2G\,\delta g + 2H\,\delta h)\,\Delta x\,\Delta y\,\Delta z.$$

Hence, if δW be the work for the entire body, we have

$$\delta W = \iiint (A\delta a + B\delta b + C\delta c + 2F\delta f + 2G\delta g + 2H\delta h)\, dx\, dy\, dz, \quad (2)$$

n which the triple integral is taken for all points in the ody.

We may in general suppose that δa has the same sign as he corresponding stress A, *i. e.* that when A is a *traction* the dge Δx is *elongated*, and when A is a *pressure* Δx is *con-racted;* accordingly, $A\delta a$ is positive; and similarly for the ther terms.

The work done by a shearing stress can also be readily leduced from the principle proved in Art. 12, and in Ex. 4, Art. 54. Thus, instead of the shearing stress G we may ake equal and opposite normal stresses ($+ G$ and $- G$) in the lanes which bisect the angles between the planes of the iven shear G. Likewise for the small shear $2\delta g$ we may ubstitute the corresponding normal elongations ($+\delta g$ and $-\delta g$) erpendicular to the bisecting planes. Hence, by super-osition, we see from (1) that the work done is equivalent to

$$G\, \delta g\, dxdydz + (-G)(-\delta g)\, dxdydz, \text{ or } 2G\, \delta g\, dxdydz,$$

s before.

57. **Verification by Green's Theorem.**—The result n (1) can be readily verified by aid of Green's general theorem *Integral Calculus*, Art. 226). For, as previously observed, n a body in equilibrium in a state of strain there must be quilibrium between the whole system of external and internal orces which act on the body; consequently, for any inde-nitely small displacements the work done by *all* the external orces must be equal to that of the internal forces of elasticity. gain, let δu, δv, δw denote the small components of displace-ent at any point xyz; then the total work done by the *ody* forces is represented by

$$\iiint \rho\, (X\delta u + Y\delta y + Z\delta w)\, dxdydz.$$

Also, if P, Q, R be the components per unit area of the xternal stress on the element dS of the surface of the body,

the total work done by the external system of stresses is represented by

$$\iint (P\delta u + Q\delta v + R\delta w)\, dS$$

taken over the entire surface.

Accordingly, if δW be the corresponding elementary work performed against the forces of elasticity of the body, we may write

$$\begin{aligned}\delta W = \iiint (X\delta u + Y\delta v + Z\delta w)\, dxdydz \\ + \iint (P\delta u + Q\delta v + R\delta w)\, dS. \qquad (3)\end{aligned}$$

Again, by equations (3) of Art. 35,

$$\begin{aligned}&\iint (P\delta u + Q\delta v + R\delta w)\, dS \\ &= \iint \{ l(A\delta u + H\delta v + G\delta w) \\ &+ m(H\delta u + B\delta v + F\delta w) \\ &+ n(F\delta u + G\delta v + C\delta w)\}\, dS,\end{aligned}$$

where lmn are the direction cosines of the *outward* drawn normal to the bounding surface at dS.

Now, by Green's theorem we have, when α, β, γ are any continuous functions of x, y, z,

$$\iiint \left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) dxdydz = \iint (l\alpha + m\beta + n\gamma)\, dS, \qquad (4)$$

where the triple integral is taken throughout any volume, and the double integral is taken over the boundary of the volume.

Consequently,

$$\begin{aligned}&\iint (P\delta u + Q\delta v + R\delta w)\, dS \\ &+ \iiint \left\{ \frac{d}{dx}(A\delta u + H\delta v + G\delta w) + \frac{d}{dy}(H\delta u + B\delta v + F\delta w) \right. \\ &\left. + \frac{d}{dz}(F\delta u + G\delta v + C\delta w) \right\} dxdydz.\end{aligned}$$

If we differentiate these expressions in the triple integral, and observe that

$$\frac{dA}{dx} + \frac{dH}{dy} + \frac{dG}{dz} + \rho X = 0, \text{ \&c.,}$$

we readily see that equation (2) becomes

$$\delta W = \iiint \left\{ A \frac{d\delta u}{dx} + B \frac{d\delta v}{dy} + C \frac{d\delta w}{dz} + F\left(\frac{d\delta w}{dy} + \frac{d\delta v}{dz}\right) \right.$$
$$\left. + G\left(\frac{d\delta u}{dz} + \frac{d\delta w}{dx}\right) + H\left(\frac{d\delta v}{dx} + \frac{d\delta u}{dy}\right) \right\} dxdydz, \qquad (5)$$

but

$$\delta a = \frac{d\delta u}{dx}, \text{ \&c., } 2\delta f = \frac{d\delta w}{dy} + \frac{d\delta v}{dz}, \text{ \&c.}$$

Accordingly we get, as before,

$$\delta W = \iiint (A\delta a + B\delta b + C\delta c + 2F\delta f + 2G\delta g + 2H\delta h)\, dxdydz.$$

58. **Potential Energy of a Strain.**—It is evident that an elastic body in a given state of strain is capable of doing a certain amount of work against external resistances. According to ordinary mechanical principles this is called the potential energy of the strain.

Since the potential energy of a perfectly elastic body in its natural state is zero, it is plain that its potential energy for any state of strain is a *positive* quantity; being equal to the amount of external work required to bring the body from its natural state to the state of strain in question. Again, for a body in equilibrium, not acted on by external force, the corresponding potential energy must be zero; and we infer that the strain at each point must be also zero. For, by the preceding, if any portion of the body be in a state of strain, that portion must possess *positive* energy—but this cannot be the case if the whole potential energy of the body is zero.

Also, it is easily seen that we may assume that the whole external work required to bring a solid from one state of strain to another is independent of the intermediate states through which the solid passes in the process. (Compare *Dynamics*, Arts. 282, &c.)

Again, if we suppose that $a'b'c'f'g'h'$ denote the initial constituents of the strain at a point xyz, and imagine, without change of temperature, the strain to be gradually altered, until it finally is represented by $abcfgh$ at the same point; then, by (3), we see that the total work performed during this change, *i.e.* the change in the potential energy, is represented by

$$\iiint(A\delta a + B\delta b + C\delta c + 2F\delta f + 2G\delta g + 2H\delta h)\,dxdydz$$

taken between the assigned limits.

In this we neglect all vibratory motion, and everything of the nature of kinetic energy, and we infer that the change in the potential energy must be a function solely of the initial and the final conditions of strain of the body. This requires that the expression

$$A\delta a + B\delta b + C\delta c + 2F\delta f + 2G\delta g + 2H\delta h$$

should be the exact variation of some function of the strain $(abcfgh)$; for, otherwise, the work in the passage from one state of strain to another would not be a function of the initial and final strains solely.

Accordingly we assume

$$A\delta a + B\delta b + C\delta c + 2F\delta f + 2G\delta g + 2H\delta h = \delta V, \qquad (6)$$

where V is a function of $abcfgh$, the strain constituents. Hence we get

$$A = \frac{dV}{da},\ B = \frac{dV}{db},\ C = \frac{dV}{dc},\ 2F = \frac{dV}{df},\ 2G = \frac{dV}{dg},\ 2H = \frac{dV}{dh}. \qquad (7)$$

Again, we may write

$$dV = Ada + Bdb + Cdc + 2Fdf + 2Gdg + 2Hdh. \qquad (8)$$

And hence, by integration,

$$V = \int_{0,0,0,0,0,0}^{a,b,c,f,g,h}(Ada + Bdb + Cdc + 2Fdf + 2Gdg + 2Hdh), \qquad (9)$$

in which we suppose that initially there is no strain.

Also, equation (2) becomes

$$\delta W = \iiint \delta V dxdydz. \qquad (10)$$

59. **Stress in terms of Strain.**—In the case of a perfectly elastic solid in equilibrium, let *abc fgh* be the constituents of the strain at any point, and *ABC FGH* the corresponding constituents of stress. Also let A', B', C', F', G', H' represent the stresses corresponding to the strains a', b', c', f', g', h'. Then, by the principle of superposition, we assume, within the limits of perfect elasticity, that if, when the stresses are added, term by term, so as to give as constituents $A + A'$, $B + B'$, &c., $H + H'$, the corresponding strain will be represented by $a + a'$, $b + b'$, ... $h + h'$. Hence it follows that if the stresses be doubled, the corresponding strains will be also doubled: and generally, if the stresses be all altered in any ratio, the corresponding strains will be altered in the same ratio. From this it follows that the constituents of stress are *linear* functions of those of the corresponding strain. Again, the strain constituents can conversely be expressed as linear functions of the corresponding stress constituents. Hence we see that if the stresses all vanish, so also must the strains: this is in conformity with the principle assumed in Art. 58. These statements have been verified by experiment for perfectly elastic solids, and are a generalization of the law originally established experimentally by Hooke, and which is usually called Hooke's law of elasticity.

Hooke, in 1676, in his *Description of Helioscopes*, published his law of elasticity under the form of the anagram, *ceiiinosssttuu*. Of this two years subsequently (in his lectures *de Potentia Restitutiva*) he stated that his anagram meant, *Ut tensio sic vis.*

60. **Strain Potential.—Elastic Coefficients.**—From (7) we infer that V must be a quadratic function of the strain, and in its general shape we may write

$$\begin{aligned}2V = {} & \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2 + \lambda_4 f^2 + \lambda_5 g^2 + \lambda_6 h^2 \\ & + 2\mu_1 bc + 2\mu_2 ca + 2\mu_3 ab + 2\mu_4 gh + 2\mu_5 hf + 2\mu_6 fg \\ & + 2\tau_1 af + 2\tau_2 ag + 2\tau_3 ah + 2\tau_4 bf + 2\tau_5 bg + 2\tau_6 bh \\ & + 2\tau_7 cf + 2\tau_8 cg + 2\tau_9 ch. \qquad (11)\end{aligned}$$

For any homogeneous substance the 21 coefficients in this expression are called the elastic coefficients of the substance, and their values depend on its elastic properties. The corresponding expressions for A, B, C, F, G, H are, by (7), the differential coefficients of this expression, taken with respect to a, b, c, &c.

EXAMPLES.

1. Find the form of the strain potential for a substance which is symmetrical with respect to a plane.

Let the plane be taken as that of xy, then V is unaltered in this case by the change of z into $-z$: this changes w into $-w$, and hence f and g change signs. Accordingly we readily get

$$2V = \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2 + \lambda_4 f^2 + \lambda_5 g^2 + \lambda_6 h^2 + 2\mu_1 bc + 2\mu_2 ca + 2\mu_3 ab + 2\mu_6 fg + 3\tau_3 ah + 2\tau_6 bh + 2\tau_9 ch.$$

The expressions for the corresponding stresses are obtained by differentiation.

2. Find the form of V if the substance be symmetrical with regard to two rectangular planes.

Let the second plane be that of xz; then, in addition to above, the change of y into $-y$ must not alter V; hence we get

$$2V = \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2 + \lambda_4 f^2 + \lambda_5 g^2 + \lambda_6 h^2 + 2\mu_1 bc + 2\mu_2 ca + 2\mu_3 ab.$$

3. Find V for a substance which is symmetrical about the axis of z.

In this case V is unaltered if we interchange the axes of x and y, *i.e.* if we interchange a with b, and f with g. Hence, remembering that $a + b$ and $ab - h^2$ are invariants in this case, we see that $2V$ is of the form

$$2V = n_1 (a+b)^2 + n_2 c^2 + n_3 (f^2 + g^2) + n_4 (h^2 - ab) + 2n_5 c (a+b).$$

4. Hence, find the form of the strain potential for an isotropic body.

In this case V must be symmetrical with respect to a, b and c; and also with respect to f, g and h. Hence we readily get

$$2V = n_1 (a+b+c)^2 + m_1 (f^2 + g^2 + h^2 - ab - bc - ac).$$

This result will be established subsequently in a different manner. The physical signification of the coefficients will be shown in Art. 64.

61. **Determinateness of Strain.**—For an elastic solid when the external forces, both bodily and superficial, are all given, the state of strain is *determinate* and *unique*, *i. e.* there cannot be two systems of strain corresponding to the same system of external forces.

This result readily appears from the method of superposition. For, if possible, let the strain systems $abcfgh$ and $a'b'c'f'g'h'$ correspond to one and the same system of external forces. In the second system if all the forces be reversed, then $-a', -b', -c', -f', -g', -h'$, will represent the corresponding system of strains. Now, if the two systems be superposed, the external forces all disappear, and the corresponding strains will be $a'-a$, $b'-b$, $c'-c$, &c.; but the potential work (Art. 57) of the stress is equal to that of the external forces; and this in the case supposed must be zero. Hence the potential energy of the stress is also zero, and from Art. 58, we infer that the potential energy of the stress for each element of the body must also be zero; consequently we must have at each point

$$a=a',\ b=b',\ c=c',\ f=f',\ g=g',\ h=h',$$

as otherwise the body would possess positive potential energy. From this it follows that if in any manner we can find one system of displacements which satisfy all the conditions of any given problem, it furnishes a solution of the problem, and is its only solution.

62. **Isotropic Bodies.**—A substance is said to be isotropic when equal and similar portions of the body, however situated in the body, possess identical elastic properties. These properties are, accordingly, independent of direction. We proceed to consider the connection between the stress and strain for such a substance.

SECTION II.—*Case of Isotropic Substances.*

63. **Expressions for Stress in an Isotropic Substance.**—Adopting Cauchy's method we assume that in an isotropic substance the three principal stresses, about which the other stresses are symmetrically distributed, have the same directions as the three principal strains, about which the others are symmetrical. Hence, as the stresses are linear functions of the strains, we may write

$$\begin{aligned} E_1 &= \lambda_1 e_1 + \lambda_2 (e_2 + e_3) \\ &= (\lambda_1 - \lambda_2)\, e_1 + \lambda_2 (e_1 + e_2 + e_3). \end{aligned}$$

Hence, altering the notation slightly, we may write

$$E_1 = \lambda\Delta + 2\mu e_1, \quad E_2 = \lambda\Delta + 2\mu e_2, \quad E_3 = \lambda\Delta + 2\mu e_3. \quad (1)$$

Again, if ξ_1, η_1, ζ_1 be the coordinates of any point relative to the principal axes of elasticity, we have

$$E_1\xi_1^2 + E_2\eta_1^2 + E_3\zeta_1^2 = \lambda\Delta(\xi_1^2 + \eta_1^2 + \zeta_1^2) + 2\mu(e_1\xi_1^2 + e_2\eta_1^2 + e_3\zeta_1^2).$$

If we now suppose the axes transformed to any new rectangular system, this equation, by Articles 10 and 67, transforms into

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2F\eta\zeta + 2G\zeta\xi + 2H\xi\eta$$
$$= \lambda\Delta(\xi^2 + \eta^2 + \zeta^2) + 2\mu(a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta).$$

Accordingly, equating coefficients, we get

$$\left.\begin{array}{l} A = \lambda\Delta + 2\mu a, \; B = \lambda\Delta + 2\mu b, \; C = \lambda\Delta + 2\mu c \\ F = 2\mu f, \quad G = 2\mu g, \quad H = 2\mu h \end{array}\right\}, \qquad (2)$$

where λ and μ are coefficients which depend on the elastic properties of the substance.

64. **Moduli of Elasticity and Rigidity.**—We can readily give a physical meaning to the coefficients λ, μ. For suppose $A = B = C$, then we get

$$a = b = c = \frac{\Delta}{3}, \text{ and } A = (\lambda + \tfrac{2}{3}\mu)\Delta. \qquad (3)$$

But this case corresponds to a uniform normal stress arising from a uniform compression or expansion; hence in this case, within the limits of elasticity, the dilatation of an isotropic substance varies with the uniform normal traction or pressure. If Π represent the uniform normal stress we may write

$$\Pi = \kappa\Delta. \qquad (4)$$

The coefficient κ is called the modulus of elasticity; or the modulus of incompressibility. This modulus can be determined experimentally, and tabulated for different isotropic substances.

If we compare with (3) we get

$$\lambda + \tfrac{2}{3}\mu = \kappa, \text{ or } \lambda = \kappa - \tfrac{2}{3}\mu. \tag{5}$$

Again, from the equations

$$F = 2\mu f, \text{ \&c.},$$

we see that μ represents the ratio of the shearing stress on any plane to the corresponding shear. This ratio is called the modulus of rigidity of the substance.

65. **Young's Modulus.**—If we suppose the stress A to be the only stress acting on the body, we have

$$\left.\begin{aligned}(\kappa - \tfrac{2}{3}\mu)\,\Delta + 2\mu a &= A\\ (\kappa - \tfrac{2}{3}\mu)\,\Delta + 2\mu b &= 0\\ (\kappa - \tfrac{2}{3}\mu)\,\Delta + 2\mu c &= 0\end{aligned}\right\}. \tag{6}$$

These give

$$A = 3\kappa\Delta, \text{ hence } a = \left(\frac{\kappa}{\mu} + \tfrac{1}{3}\right)\Delta;$$

therefore
$$A = \frac{9\kappa\mu}{3\kappa + \mu}\,a. \tag{7}$$

This shows that the elongation is proportional to the stress.

The coefficient of a in (7) is called Young's modulus of elasticity. If this modulus be represented by E, we have

$$E = \frac{9\kappa\mu}{3\kappa + \mu}. \tag{8}$$

In this case it should be observed that when a is positive, the lateral strains b and c are in general negative; they are sometimes written

$$b = c = -\eta a,$$

where
$$\eta = \tfrac{1}{2}\,\frac{3\kappa - 2\mu}{(3\kappa + \mu)}. \tag{9}$$

66. Equations deduced from Potential Energy.— The equations connecting stress and strain for isotropic substances can also be immediately deduced from the potential V. For, in the case of isotropic bodies it is manifest that the function V must be independent of the directions of the coordinate axes, and consequently it must be a function of the *invariants* of the strain; hence, since V is a quadratic function, we may assume

$$V = \kappa_1 (a+b+c)^2 + \kappa_2 (f^2 + g^2 + h^2 - ab - ac - bc). \quad (10)$$

By differentiation we get

$$A = \frac{dV}{da} = 2\kappa_1 \Delta - \kappa_2 (b+c), \text{ \&c.}, \; F = \kappa_2 f, \text{ \&c.}$$

These equations can be immediately reduced to the form given in (2).

Again, if in (2) we substitute for a, b, c, &c. their values given in Art. 21, we get

$$\left.\begin{aligned} &A = (\kappa - \tfrac{2}{3}\mu)\Delta + 2\mu \frac{du}{dx}, \; B = (\kappa - \tfrac{2}{3}\mu)\Delta + 2\mu \frac{dv}{dy}, \\ &C = (\kappa - \tfrac{2}{3}\mu)\Delta + 2\mu \frac{dw}{dz}, \; F = \mu \left(\frac{dw}{dy} + \frac{dv}{dz}\right), \\ &G = \mu \left(\frac{dw}{dx} + \frac{du}{dz}\right), \quad H = \mu \left(\frac{du}{dy} + \frac{dv}{dx}\right) \end{aligned}\right\}, \quad (11)$$

where $$\Delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

From these equations we infer that whenever the expressions for the displacements u, v, w are known, the corresponding state of stress for an isotropic body can be at once determined, provided its coefficients of rigidity and of incompressibility are known.

In general the external surface stresses are supposed given, and also the external body forces; and the problem for solution is the determination of the resulting displacements u, v, w at each point in the body.

In many problems the body forces are so small in comparison with the stresses that they may be neglected.

For a pure strain equations (11) become

$$\left.\begin{aligned} A &= \lambda\Delta + 2\mu\frac{d^2\phi}{dx^2},\ B = \lambda\Delta + 2\mu\frac{d^2\phi}{dy^2},\ C = \lambda\Delta + 2\mu\frac{d^2\phi}{dz^2} \\ F &= 2\mu\frac{d^2\phi}{dydz},\ G = 2\mu\frac{d^2\phi}{dx\,dz},\ H = 2\mu\frac{d^2\phi}{dxdy} \end{aligned}\right\}. \quad (12)$$

67. **Spherical Pure Strain.—Cylindrical Pure Strain.**—In the case of a pure spherical strain, if P, Q, R be the principal stresses at any point, we get from (27), Art. 26,

$$\left.\begin{aligned} P &= (\kappa - \tfrac{2}{3}\mu)\Delta + 2\mu\frac{d^2\phi}{dr^2},\ Q = R = (\kappa - \tfrac{2}{3}\mu)\Delta + \frac{2\mu}{r}\frac{d\phi}{dr} \\ \Delta &= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) \end{aligned}\right\}. \quad (13)$$

Also, for a cylindrical pure strain the principal stresses, by Art. 27, are

$$\left.\begin{aligned} P &= (\kappa - \tfrac{2}{3}\mu)\Delta + 2\mu\frac{d^2\phi}{dr^2},\ Q = (\kappa - \tfrac{2}{3}\mu)\Delta + \frac{2\mu}{r}\frac{d\phi}{dr} \\ R &= (\kappa - \tfrac{2}{3}\mu)\Delta, \qquad \Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) \end{aligned}\right\}. \quad (14)$$

Of these P is a *radial* stress, while Q and R are at right angles to the radius vector. Hence they are called respectively the *radial,* the *circular*, and the *longitudinal* stresses at the point.

68. **Equations of Equilibrium.**—If we substitute in the general equations of Art. 52, we readily obtain

$$\left.\begin{aligned} (\kappa + \tfrac{1}{3}\mu)\frac{d\Delta}{dx} + \mu\nabla^2 u + \rho X = 0 \\ (\kappa + \tfrac{1}{3}\mu)\frac{d\Delta}{dy} + \mu\nabla^2 v + \rho Y = 0 \\ (\kappa + \tfrac{1}{3}\mu)\frac{d\Delta}{dz} + \mu\nabla^2 w + \rho Z = 0 \end{aligned}\right\}, \quad (15)$$

where $$\nabla^2 u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}.$$

Also, since

$$\frac{du}{dz} - \frac{dw}{dx} = 2\theta_2, \text{ \&c.},$$

we easily see that

$$\nabla^2 u = \frac{d\Delta}{dx} + 2\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right),$$

and accordingly equations (15) may be written

$$\left.\begin{aligned}(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dx} + 2\mu\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right) + \rho X = 0\\(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dy} + 2\mu\left(\frac{d\theta_3}{dx} - \frac{d\theta_1}{dz}\right) + \rho Y = 0\\(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dz} + 2\mu\left(\frac{d\theta_1}{dy} - \frac{d\theta_2}{dx}\right) + \rho Z = 0\end{aligned}\right\}. \qquad (16)$$

Again, in a pure strain, these become

$$\left.\begin{aligned}(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dx} + \rho X = 0\\(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dy} + \rho Y = 0\\(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{dz} + \rho Z = 0\end{aligned}\right\}. \qquad (17)$$

These at once lead to the equation

$$(\kappa + \tfrac{4}{3}\mu)\, d\Delta + \rho\,(X dx + Y dy + Z dz) = 0. \qquad (18)$$

Hence for a conservative system of forces we have

$$(\kappa + \tfrac{4}{3}\mu)\,\Delta + \rho U = \text{const.},$$

where $$X dx + Y dy + Z dz = dU. \qquad (19)$$

In a pure strain, if there be no external bodily forces we have $X = 0$, $Y = 0$, $Z = 0$, and accordingly

$$\Delta = \text{const.} \qquad (20)$$

Hence, in such case the condensation produced by the strain is the same at all points of the body.

When the forces satisfy the equation

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

we immediately see from equations (16) that in all cases of strain we have

$$\nabla^2 \Delta = 0. \qquad (21)$$

Accordingly the condensation Δ is in all such cases the sum of a number of *harmonic functions* (*Differential Calculus*, Art. 333).

69. **Equations of Equilibrium in Curvilinear Coordinates.**—Adopting the notation of Art. 29, and multiplying equations (16) by l_1, m_1, n_1, respectively, we get by addition

$$(\kappa + \tfrac{4}{3}\mu)\frac{d\Delta}{ds_1} + \rho\,(l_1X + m_1Y + n_1Z)$$

$$+ 2\mu\left\{l_1\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right) + m_1\left(\frac{d\theta_3}{dx} - \frac{d\theta_1}{dz}\right) + n_1\left(\frac{d\theta_1}{dy} - \frac{d\theta_2}{dx}\right)\right\} = 0. \qquad (22)$$

Again, by Art. 39, we have

$$l_1\left(\frac{d\theta_2}{dz} - \frac{d\theta_3}{dy}\right) + m_1\left(\frac{d\theta_3}{dx} - \frac{d\theta_1}{dz}\right) + n_1\left(\frac{d\theta_1}{dy} - \frac{d\theta_2}{dx}\right)$$

$$= l_2\frac{d\theta_1}{ds_3} + m_2\frac{d\theta_2}{ds_3} + n_2\frac{d\theta_3}{ds_3} - l_3\frac{d\theta_1}{ds_2} - m_3\frac{d\theta_2}{ds_2} - n_3\frac{d\theta_3}{ds_2}.$$

Hence, proceeding as in Art. 40, we find immediately that the latter expression is equal to

$$\frac{d\Theta_2}{ds_3} - \frac{d\Theta_3}{ds_2} + \frac{\Theta_3}{{}_3R_2} - \frac{\Theta_2}{{}_2R_3}$$

$$= \frac{1}{k_1k_2}\left(\frac{d(\Theta_2k_2)}{d\rho_3} - \frac{d(\Theta_3k_3)}{d\rho_2}\right).$$

Finally, if Ξ_1, Ξ_2, Ξ_3 be the components of the external force in the directions ds_1, ds_2, ds_3, respectively, we get, by equation (22),

$$\left.\begin{aligned}(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{ds_1}+\frac{2\mu}{k_2k_3}\left(\frac{d(\Theta_2k_2)}{d\rho_3}-\frac{d(\Theta_3k_3)}{d\rho_2}\right)+\rho\,\Xi_1=0,\\(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{ds_2}+\frac{2\mu}{k_3k_1}\left(\frac{d(\Theta_3k_3)}{d\rho_1}-\frac{d(\Theta_1k_1)}{d\rho_3}\right)+\rho\,\Xi_2=0,\\(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{ds_3}+\frac{2\mu}{k_1k_2}\left(\frac{d(\Theta_1k_1)}{d\rho_2}-\frac{d(\Theta_2k_2)}{d\rho_1}\right)+\rho\,\Xi_3=0\end{aligned}\right\}. \quad (23)$$

In the case of polar coordinates these become

$$(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{dr}+\frac{2\mu}{r\sin\theta}\left(\frac{d(\Theta_2)}{d\phi}-\frac{d(\Theta_3\sin\theta)}{d\theta}\right)+\rho\,\Xi_1=0,$$

$$(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{rd\theta}+\frac{2\mu}{r}\left(\frac{d(\Theta_3r)}{dr}-\frac{d\Theta_1}{\sin\theta d\phi}\right)+\rho\,\Xi_2=0.$$

$$(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{r\sin\theta d\phi}+\frac{2\mu}{r}\left(\frac{d\Theta_1}{d\theta}-\frac{d(\Theta_2r)}{dr}\right)+\rho\,\Xi_3=0.$$

70. **Resilience.**—The elasticity of every solid is sensibly perfect up to a certain limit, which is called the limit of perfect elasticity for the substance. When strained beyond this limit the body receives a *set*, and in some cases is *broken*. Experience shows that the safe working stress of any material should lie within its limits of elasticity. For a given species of strain the amount of work performed on a substance in straining it to the corresponding elastic limits is called the elastic resilience of the substance for the strain. Such resilience can for particular substances be calculated and estimated in foot pounds similarly to any other kind of work.

Section III.—*Applications.*

71. **Spherical Shell under Uniform Normal Pressure.**—An isotropic solid, whose boundaries are concentric spherical surfaces, is acted on by uniform pressures, p and p', on its outer and inner surfaces solely, to find its condition of stress and strain at each point.

In this case it is immediately seen that the strain is *pure*, also since there are no *body* forces, we have, by (20), Art. 68,

$$\Delta = \text{constant} = a \text{ (suppose)}. \tag{1}$$

Hence, by (13), Art. 67, we have

$$\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = ar^2;$$

therefore

$$\frac{d\phi}{dr} = \frac{ar}{3} + \frac{\beta}{r^2}, \tag{2}$$

in which a and β are small constants, whose values are determined by aid of the boundary conditions.

First, let the sphere be solid, *i. e.* let there be no inner boundary. In this case we must have $\beta = 0$, as otherwise the displacement at the centre would become infinite.

Hence, we have

$$\frac{d\phi}{dr} = \frac{ar}{3}, \quad \frac{d^2\phi}{dr^2} = \frac{a}{3}.$$

Accordingly, from (13) the principal stresses at any point for a solid sphere are given by

$$P = Q = R = \kappa a.$$

Hence in this case the sphere is subject to uniform compression, as might have been readily foreseen.

The value of a is determined from the given external pressure.

In the case of a shell, we get

$$\frac{d^2\phi}{dr^2} = \frac{a}{3} - \frac{2\beta}{r^3}, \tag{3}$$

and accordingly at any point we have

$$\left.\begin{aligned} P &= \kappa a - \frac{4\mu\beta}{r^3} \\ Q = R &= \kappa a + \frac{2\mu\beta}{r^3} \end{aligned}\right\}. \tag{4}$$

Now, suppose a and b to be the radii of the outer and inner surfaces, respectively; then, since for the outer surface $P = -p$, and for the inner $P = -p'$, we get

$$\left.\begin{aligned} -p &= \kappa a - \frac{4\mu\beta}{a^3} \\ -p' &= \kappa a - \frac{4\mu\beta}{b^3} \end{aligned}\right\}. \qquad (5)$$

These equations enable us to determine a and β in terms of the given pressures p and p'. Also the principal stresses at any distance r from the centre are given by

$$P = -\frac{a^3b^3}{a^3 - b^3}\left\{p\left(\frac{1}{b^3} - \frac{1}{r^3}\right) + p'\left(\frac{1}{r^3} - \frac{1}{a^3}\right)\right\}. \qquad (6)$$

$$Q = R = -\frac{a^3b^3}{a^3 - b^3}\left\{p\left(\frac{1}{b^3} + \frac{1}{2r^3}\right) - p'\left(\frac{1}{a^3} + \frac{1}{2r^3}\right)\right\}. \qquad (7)$$

It may be here observed that if all the conditions of any problem are satisfied by assuming the strain to be pure, then by Art. 61 the only solution of the problem is one for which the strain is pure.

EXAMPLES.

1. If the shell be subject to interior pressure only, find the stress at any point within the shell.

Ans. $P = -p'\dfrac{b^3}{r^3}\dfrac{a^3 - r^3}{a^3 - b^3}$, $\quad Q = p'\dfrac{b^3}{a^3 - b^3}\left(1 + \frac{1}{2}\dfrac{a^3}{r^3}\right)$.

2. In Ex. 1 show that if the inner pressure be increased indefinitely, the rupture of the shell will take the form of transverse stretching at the inner surface.

Here Q is greater than $-P$, and the greatest value of Q is at the inner surface, &c.

3. For a thin shell, whose thickness is t, show that the value of the transverse stress Q at the inner surface is $p'\dfrac{b}{2t}$, approximately.

72. Hollow Cylinder under Uniform Pressure.—We shall next investigate the distribution of strain for a hollow circular cylinder of indefinite length under the action of uniform normal pressures, p and p', on its outer and inner surfaces, solely.

Take the axis of the cylinder as that of z, and it is evident that the displacement of each point is parallel to the plane of

xy; and also that the strain is pure. Hence, as before, we must have $\Delta = a$, a constant. Accordingly, by Art. 67, we have

therefore
$$\left.\begin{aligned} \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) &= ar, \\ \frac{d\phi}{dr} &= \frac{a}{2}r + \frac{\beta}{r} \end{aligned}\right\}, \tag{8}$$

where a and β are constants, to be determined by aid of the given pressures.

Again, the principal stresses at any distance are immediately found from Art. 67 to be

$$\left.\begin{aligned} P &= (\kappa + \tfrac{1}{3}\mu)\,a - 2\mu\frac{\beta}{r^2} \\ Q &= (\kappa + \tfrac{1}{3}\mu)\,a + 2\mu\frac{\beta}{r^2} \\ R &= (\kappa - \tfrac{2}{3}\mu)\,a \end{aligned}\right\}, \tag{9}$$

where P is the *radial*, Q the *circular*, and R the *longitudinal* stress. The latter stress is constant, and is parallel to the axis of the cylinder. Q is often styled the *hoop stress* of the cylinder.

Again, we have

$$-p = (\kappa + \tfrac{1}{3}\mu)\,a - 2\frac{\mu\beta}{a^2}, \quad -p' = (\kappa + \tfrac{1}{3}\mu)\,a - 2\mu\frac{\beta}{b^2}, \tag{10}$$

where a and b are the radii of the external and internal surfaces.

If we eliminate a and β between these and the equations in (9), we get

$$\left.\begin{aligned} P &= -\frac{a^2b^2}{a^2-b^2}\left\{p\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + p'\left(\frac{1}{r^2} - \frac{1}{a^2}\right)\right\} \\ Q &= -\frac{a^2b^2}{a^2-b^2}\left\{p\left(\frac{1}{b^2} + \frac{1}{r^2}\right) - p'\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right\} \\ R &= -\frac{3\kappa - 2\mu}{3\kappa + \mu}\cdot\frac{a^2p - b^2p'}{a^2 - b^2} \end{aligned}\right\}. \tag{11}$$

For a cylinder subject to internal pressure only these become

$$P = -p'\frac{b^2}{r^2}\frac{a^2 - r^2}{a^2 - b^2}, \quad Q = p'\frac{b^2}{r^2}\frac{a^2 + r^2}{a^2 - b^2}, \quad R = \frac{3\kappa - 2\mu}{3\kappa + \mu}\frac{b^2 p'}{a^2 - b^2}. \quad (12)$$

For a very thin cylinder the values of Q and R become very great in general. Thus, if t be the thickness of the cylinder, the tearing stress Q becomes, approximately, equal to $p'\frac{a}{t}$; also $R = \frac{1}{2}Q$, approximately.

From the foregoing we see that the radial stress is always a *pressure*, while the directions of the other principal stresses depend on the relative magnitudes of p, p', a and b. For instance, if $\frac{p'}{p} < \frac{2a^2}{a^2 + b^2}$ the stress Q is a *pressure*; if $\frac{p}{p'} < \frac{2b^2}{a^2 + b^2}$ Q is a *traction* everywhere. Again, Q vanishes at a distance r, where $r = ab\sqrt{\frac{p' - p}{a^2 p - b^2 p'}}$.

The method given above is approximately applicable to cylindrical boilers under steam pressure and leads to important practical results, especially in connection with the maximum pressure consistent with the *safety* of the boiler.

Examples.

1. Find the condition that the stress shall be two dimensional.

Ans. $R = 0$, or $\frac{p}{b^2} = \frac{p'}{a^2}$.

2. If T be the greatest stress consistent with perfect elasticity of the material of a cylindrical tube, find the condition for the elastic safety of the tube.

Ans. $\frac{a^2}{b^2} > \frac{p' + T}{2p + T - p'}$. Lamè's *Leçons de l'élasticité*, p. 190.

3. If f be the greatest stress a thin hollow cylinder can bear without rupture, prove that the internal pressure which would burst it is equal to λf, where λ is the ratio of the thickness of the cylinder to its radius.

In steam boilers it is considered prudent to make the working pressure only one-eighth of the bursting pressure. That is, 8 is taken as the factor of safety in this case.

4. Show that for the same thickness a spherical shell is twice as strong as a cylindrical shell of equal radius.

73. **Solid Sphere subject to Self Attraction.**—Next let us suppose a solid sphere subject solely to its own attraction, and such that the sphere is homogeneous when free from strain. In this case if V be the potential of attraction, and if ρ be the constant density, and $\rho(1-\Delta)$ the strained density, then from (17), Art. 68, we immediately get

$$(\kappa+\tfrac{4}{3}\mu)\frac{d\Delta}{dr}+\rho\,(1-\Delta)\frac{dV}{dr}=0. \tag{13}$$

Now, observing that all our equations are based on the assumption that the strains at each point are *very small*, and that in these equations *small quantities of the second order are neglected*, it follows that Δ must be a very small quantity in (13), and hence that the term $\rho\Delta\dfrac{dV}{dr}$ may be neglected, being very small in comparison with $\rho\dfrac{dV}{dr}$.

Again, $\dfrac{dV}{dr}$ is the attraction at the distance r, and hence, by dividing the sphere into concentric shells, we readily see that

$$\frac{dV}{dr}=-\frac{4\pi\gamma\rho}{r^2}\int_0^r(1-\Delta')\,r'^2dr'. \tag{14}$$

Also, Δ' must be very small at all points, and accordingly, to *the degree of approximation adopted throughout*, we may write

$$\frac{dV}{dr}=-\frac{4\pi\gamma\rho r}{3}. \tag{15}$$

If now we assume

$$c^2=\frac{3\kappa+4\mu}{4\pi^2\gamma\rho^2}, \tag{16}$$

equation (13) becomes

$$\frac{d\Delta}{dr}=\frac{r}{c^2}:$$

therefore

$$\Delta=a+\frac{r^2}{2c^2}, \tag{17}$$

in which α is a small quantity, viz. the dilatation at the centre of the sphere. Again, $\frac{r}{c}$ must also be small for all points in the substance.

Hence we have

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \alpha r^2 + \frac{r^4}{2c^2};$$

therefore

$$\frac{d\phi}{dr} = \frac{\alpha r}{3} + \frac{r^3}{10c^2} + \frac{\beta}{r^2}.$$

Here $\beta = 0$, as otherwise the displacement at the centre o the sphere would become infinite.

Consequently we have

$$\frac{d\phi}{dr} = \frac{\alpha r}{3} + \frac{r^3}{10c^2}. \tag{18}$$

Hence, by integration, we get

$$\phi = \frac{\alpha}{6}r^2 + \frac{r^4}{40c^2}. \tag{19}$$

Again, by Art. 67, the principal stresses at the distance are given by

$$\left.\begin{aligned} P &= \kappa\alpha + \frac{r^2}{2c^2}\left(\kappa + \tfrac{8}{15}\mu\right) \\ Q &= \kappa\alpha + \frac{r^2}{2c^2}\left(\kappa - \tfrac{4}{15}\mu\right) \end{aligned}\right\}, \tag{20}$$

in which α is still undetermined.

If we suppose that there is no external pressure on the surface of the sphere, we have

$$\kappa\alpha = -\frac{a^2}{2c^2}\left(\kappa + \tfrac{8}{15}\mu\right), \tag{21}$$

where a is the radius of the sphere.

This gives for the principal stresses at any internal istance r,

$$\left.\begin{aligned} P &= -\frac{a^2-r^2}{2c^2}\left(\kappa+\tfrac{8}{15}\mu\right) \\ Q &= -\frac{1}{2c^2}\left\{\kappa\,(a^2-r^2)+\tfrac{4}{15}\mu\,(2a^2-r^2)\right\} \end{aligned}\right\}. \qquad (22)$$

Also, if we substitute in (19) we have

$$\phi = \frac{1}{40}\frac{r^4}{c^2} - \frac{r^2a^2}{12c^2}\left(1+\tfrac{8}{15}\frac{\mu}{\kappa}\right). \qquad (23)$$

The displacements of any point are obtained at once from , by differentiation.

Another mode of discussion of this problem will be found ı Professor Minchin's *Treatise on Statics*, vol. ii., pp. 456–461.

More generally, let the sphere be subject to a uniform xternal pressure p, to find the principal stresses at any oint. In this case we have

$$-p = \kappa a + \frac{a^2}{2c^2}\left(\kappa+\tfrac{8}{15}\mu\right).$$

Ience
$$P = -p - \frac{a^2-r^2}{2c^2}\left(\kappa+\tfrac{8}{15}\mu\right). \qquad (24)$$

EXAMPLE.

1. Show from the principle of dimensions in physical units, that c is a linear agnitude; and also that $\frac{r}{c}$ is a small quantity.

74. **Circular Cylinder under Uniform Tangential tress.**—An isotropic elastic substance is bounded by two)axal circular cylinders, of indefinite length, and has its ıner surface rigidly fixed, while its outer surface is subjected) a uniform *tangential* traction, whose direction at each point , at right angles to the axis of the cylinder, to find the strain nd stress at each point.

Here, from the conditions of the problem we may assume that the displacement at any point P is at right angles to the axis and also to the perpendicular r drawn from P to the axis.

Hence we may write

$$u = -y\phi, \quad v = x\phi, \quad w = 0, \tag{25}$$

where ϕ is a function of r.

From these we get $\frac{du}{dx} + \frac{dv}{dy} = 0$; and consequently we have

$$\Delta = 0$$

at every point.

Again, since the external forces are wholly superficial, we get, from equations (15), Art. 68,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \quad \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0. \tag{26}$$

But $$\frac{du}{dx} = -\frac{xy}{r}\frac{d\phi}{dr}, \quad \frac{du}{dy} = -\phi - \frac{y^2}{r}\frac{d\phi}{dr},$$

and the former equation in (26) leads to

$$3\frac{d\phi}{dr} + r\frac{d^2\phi}{dr^2} = 0, \text{ or } \frac{d}{dr}\left(r^3\frac{d\phi}{dr}\right) = 0.$$

Hence we have

$$\phi = \beta - \frac{a}{r^2},$$

where a and β are constants.

Let now a and b be the outer and inner radii of the cylindrical boundaries; then, from the conditions of the problem, we have $\phi = 0$ when $r = b$. This gives $\beta = \frac{a}{b^2}$, and accordingly

$$\phi = a\left(\frac{1}{b^2} - \frac{1}{r^2}\right); \tag{27}$$

therefore

$$u = -ay\left(\frac{1}{b^2} - \frac{1}{r^2}\right), \; v = ax\left(\frac{1}{b^2} - \frac{1}{r^2}\right).$$

Again, the stresses at any point are given by

$$A = 2\mu \frac{du}{dx}, \quad B = 2\mu \frac{dv}{dy}, \quad H = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right).$$

Hence we have

$$A = -B = \frac{4\mu axy}{r^4}, \; H = 2\frac{\mu a}{r^4}(y^2 - x^2).$$

These give the constituents of the stress at each point when a is known.

Again, the equation of the stress quadric

$$A\xi^2 + B\eta^2 + 2H\eta\xi = K$$

becomes in this case

$$\frac{4\mu a}{r^4}\{xy(\xi^2 - \eta^2) + (y^2 - x^2)\xi\eta\} = K,$$

or

$$4\mu\frac{a}{r^4}(x\xi + y\eta)(y\xi - x\eta) = K.$$

If now $x = r\cos\theta$, $y = r\sin\theta$, this equation becomes

$$\frac{4\mu a}{r^2}(\xi\cos\theta + \eta\sin\theta)(\xi\sin\theta - \eta\cos\theta) = K,$$

hence the stress quadric when referred to the direction of the radius r along with that of a line perpendicular to r, becomes

$$4\mu\frac{a}{r^2}\xi'\eta' = K.$$

This shows that the whole stress is equivalent to a shearing stress whose amount is represented by $2\frac{\mu a}{r^2}$.

Again, at the external surface of the cylinder this shearing stress must be equal and opposite to the external tangential shearing stress F; this gives

$$F = -2\frac{\mu a}{a^2}, \text{ or } a = -\frac{Fa^2}{2\mu}.$$

Accordingly the displacement at each point is given by the equations

$$u = \frac{Fa^2}{2\mu} y \left(\frac{1}{b^2} - \frac{1}{r^2}\right), \quad v = -\frac{Fa^2}{2\mu} x \left(\frac{1}{b^2} - \frac{1}{r^2}\right). \qquad (28)$$

The foregoing analysis shows that these displacements satisfy all the conditions of the problem, and accordingly furnish its complete solution.

We shall next solve the problem by the method of cylindrical coordinates.

Here, we have $a_1 = 0$, $w = 0$; and the equations of Art. 38 show that

$$a_2 = \phi, \quad a_1 = 0, \quad b_1 = \frac{d\phi}{d\theta}, \quad c_1 = 0, f_1 = g_1 = 0, \; 2h_1 = r\frac{d\phi}{dr}.$$

Again, as ϕ is a function of r solely, we have $b_1 = 0$; accordingly we have

$$A_1 = B_1 = C_1 = 0, \quad H_1 = \mu r \frac{d\phi}{dr}.$$

This shows that the whole stress reduces to the shearing stress $\mu r \dfrac{d\phi}{dr}$. The student can readily supply the remainder of the investigation.

Equations (28) show that if the body be supposed divided into a number of indefinitely thin cylindrical shells, having the axis of the cylinder as a common axis, then the whole motion of each cylinder consists of a circular shearing rotation; the angular motion $\Delta\theta$ for a cylinder whose radius is r being given by

$$\Delta\theta = \frac{Fb^2}{2\mu}\left(\frac{1}{b^2} - \frac{1}{r^2}\right).$$

Hence the rotation is greatest at the exterior surface, where it amounts to $\dfrac{F}{2\mu}\dfrac{a^2 - b^2}{a^2}$.

EXAMPLES.

1. In the preceding problem show that each radial line of the cylinder becomes an hyperbola after the strain.

It is easily seen that any radius becomes after strain the hyperbola represented by

$$xy = \frac{Fa^2}{2\mu}\left(1 - \frac{x^2}{b^2}\right).$$

2. In the same case find the directions of the lines of stress.

From Art. 54 it follows at once that a line of stress at any point intersects the corresponding vector r at an angle either of 45°, or of 135°; hence the lines of stress are equiangular spirals, represented by the equations $r = e^{\theta}$, and $r = e^{-\theta}$.

75. **Planetary Crust.**—We shall conclude the Chapter with the investigation of the stress in the case of a uniform planetary crust surrounding a uniform spherical nucleus, taking attraction into account.

Suppose a and b to be the outer and inner radii of the crust, ρ its density, and $\rho(1+n)$ that of the nucleus; then, within the crust, at any distance from the centre, the attraction consists of two parts, one that of a homogeneous sphere of radius a and density ρ, the other that of a sphere of radius b and density $n\rho$.

Hence, as in Art. 73, we have

$$\frac{dV}{dr} = -\tfrac{4}{3}\pi\gamma\rho\left(r + \frac{nb^3}{r^2}\right).$$

Accordingly

$$\frac{d\Delta}{dr} = \frac{r}{c^2} + \frac{nb^3}{c^2 r^2},$$

where c^2 has the same value as in Art. 73;

therefore $$\Delta = \tfrac{1}{2}\frac{r^2}{c^2} - \frac{nb^3}{c^2 r} + a.$$

Hence

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = ar^2 + \tfrac{1}{2}\frac{r^4}{c^2} - \frac{nb^3 r}{c^2};$$

therefore

$$\frac{d\phi}{dr} = \tfrac{1}{3}ar + \tfrac{1}{10}\frac{r^3}{c^2} - \frac{nb^3}{2c^2} + \frac{\beta}{r^2}, \tag{29}$$

where a and β are constants.

Proceeding, as before, we get for the radial and tangential stresses

$$\left.\begin{aligned} P &= a\kappa + \tfrac{1}{2}\frac{r^2}{c^2}\left(\kappa + \tfrac{8}{15}\mu\right) - \frac{nb^3}{c^2 r}\left(\kappa - \tfrac{2}{3}\mu\right) - \frac{4\mu\beta}{r^3} \\ Q &= a\kappa + \tfrac{1}{2}\frac{r^2}{c^2}\left(\kappa - \tfrac{4}{15}\mu\right) - \frac{nb^3}{c^2 r}\left(\kappa + \tfrac{1}{3}\mu\right) + \frac{2\mu\beta}{r^3} \end{aligned}\right\}. \qquad (30)$$

Again, the stress within the nucleus is determined as in Art. 73, provided we substitute $(1+n)\rho$ instead of ρ. Hence if c' be the new value of c, we have $c' = \dfrac{c}{1+n}$; and consequently the stresses are given by the equations

$$\left.\begin{aligned} P &= \kappa a' + \frac{r^2(1+n)^2}{2c^2}\left(\kappa + \tfrac{8}{15}\mu\right) \\ Q &= \kappa a' + \frac{r^2(1+n)^2}{2c^2}\left(\kappa - \tfrac{4}{15}\mu\right) \end{aligned}\right\}, \qquad (31)$$

assuming that κ and μ are the same for the nucleus as for the crust.

Now, at points on the inner boundary of the crust the values of P given by (30) and (31) must be equal, and the two values of the radial displacements must be also equal.

These give the equations

$$\kappa(a - a') - \frac{4\mu\beta}{b^2} = \frac{2nb^2}{c^2}\left(\kappa - \tfrac{1}{15}\mu\right) + \frac{b^2 n^2}{2c^2}\left(\kappa + \tfrac{8}{15}\mu\right), \qquad (32)$$

and

$$a - a' = \frac{3nb^2}{10c^2}(7 + n) - \frac{3\beta}{b^3}. \qquad (33)$$

Again, if p be the value of P at the outer surface of the crust, we have

$$p = a\kappa + \tfrac{1}{2}\frac{a^2}{c^2}\left(\kappa + \tfrac{8}{15}\mu\right) - \frac{nb^3}{c^2 a}\left(\kappa - \tfrac{2}{3}\mu\right) - \frac{4\mu\beta}{a^3}. \qquad (34)$$

The values of a, a', and β can be determined by means of these three equations.

Examples.

1. At any point in an isotropic substance prove that directions of equal strain are also directions of equal stress.

2. Assuming Clerk Maxwell's theory that the attraction of matter can be produced by a distribution of stress in the ether, find the constituents of the stress at any point in terms of the potential of the attraction.

Let V be the potential of the attraction, then, since in the case supposed the stresses of the ether have the same effect as the forces of gravitation, equations (1), Art. 52, at once lead to the equations

$$\frac{dA}{dx}+\frac{dH}{dy}+\frac{dG}{dz}=\rho\frac{dV}{dx},$$
$$\frac{dH}{dx}+\frac{dB}{dy}+\frac{dF}{dz}=\rho\frac{dV}{dy},$$
$$\frac{dG}{dx}+\frac{dF}{dy}+\frac{dC}{dz}=\rho\frac{dV}{dz},$$

where ρ is the density of matter at the point xyz.

Again, by Poisson's theorem, we have

$$\rho=-\frac{1}{4\pi\gamma}\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right).$$

Hence, if

$$\frac{dV}{dx}=\phi_1,\quad \frac{dV}{dy}=\phi_2,\quad \frac{dV}{dz}=\phi_3,$$

the first of the preceding equations becomes

$$\frac{dA}{dx}+\frac{dH}{dy}+\frac{dG}{dz}=-\frac{1}{4\pi\gamma}\phi_1\left(\frac{d\phi_1}{dx}+\frac{d\phi_2}{dy}+\frac{d\phi_3}{dz}\right)$$
$$=\frac{1}{8\pi\gamma}\frac{d}{dx}\left(\phi_2^2+\phi_3^2-\phi_1^2\right)-\frac{1}{4\pi\gamma}\frac{d}{dy}(\phi_1\phi_2)-\frac{1}{4\pi\gamma}\frac{d}{dz}(\phi_1\phi_3),$$

since

$$\frac{d\phi_2}{dx}=\frac{d\phi_1}{dy},\quad \frac{d\phi_3}{dx}=\frac{d\phi_1}{dz}.$$

In like manner we readily get two similar equations; and we see that the system can be satisfied by the following constituents of stress,

$$A=\frac{1}{8\pi\gamma}(\phi_2^2+\phi_3^2-\phi_1^2),\quad H=-\frac{\phi_1\phi_2}{4\pi\gamma},$$
$$B=\frac{1}{8\pi\gamma}(\phi_3^2+\phi_1^2-\phi_2^2),\quad G=-\frac{\phi_3\phi_1}{4\pi\gamma}$$
$$C=\frac{1}{8\pi\gamma}(\phi_1^2+\phi_2^2-\phi_3^2),\quad F=-\frac{\phi_2\phi_3}{4\pi\gamma}.$$

3. Find the principal stresses in the preceding system.

When referred to the principal stresses as axes of coordinates, we must have

$$F=0,\quad G=0,\quad H=0,$$

and accordingly two of the quantities ϕ_1, ϕ_2, ϕ_3 must vanish. But if $\phi_2 = 0$ and $\phi_3 = 0$, ϕ_1 becomes the resultant attraction at the point.

Hence if E_1, E_2, E_3 be the principal stresses at the point we have

$$E_1 = -\frac{R^2}{8\pi\gamma}, \quad E_2 = E_3 = \frac{R^2}{8\pi\gamma},$$

where R represents the resultant attraction at the point. Accordingly the stress of the medium consists of ***pressure along the line of resultant force, accompanied by equal tension in all directions perpendicular to the resultant force.***

4. Show that the preceding system of stress cannot be satisfied by assuming it to arise from the deformation of the ether regarded as an isotropic substance.

Adopting the usual notation, we get from equations 11, Art. 66,

$$\lambda\Delta + 2\mu\frac{du}{dx} = \frac{1}{8\pi\gamma}(\phi_2^2 + \phi_3^2 - \phi_1^2), \quad \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) = -\frac{\phi_2\phi_3}{4\pi\gamma},$$

$$\lambda\Delta + 2\mu\frac{dv}{dy} = \frac{1}{8\pi\gamma}(\phi_3^2 + \phi_1^2 - \phi_2^2), \quad \mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) = -\frac{\phi_3\phi_1}{4\pi\gamma},$$

$$\lambda\Delta + 2\mu\frac{dw}{dz} = \frac{1}{8\pi\gamma}(\phi_1^2 + \phi_2^2 - \phi_3^2), \quad \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) = -\frac{\phi_1\phi_2}{4\pi\gamma}.$$

Hence we get $\qquad (\lambda + 2\mu)\Delta = \dfrac{1}{8\pi\gamma}(\phi_1^2 + \phi_2^2 + \phi_3^2)\cdot$

Also, since $\nabla^2\phi_1 = 0$, we have

$$\nabla^2(\phi_1)^2 = 2\left\{\left(\frac{d\phi_1}{dx}\right)^2 + \left(\frac{d\phi_1}{dy}\right)^2 + \left(\frac{d\phi_1}{dz}\right)^2\right\}, \text{ \&c.}$$

Again, since equation (21), Art. 68, holds good in this case, we have also in parts of space where there is no matter

$$\nabla^2\Delta = 0, \text{ or}$$

$$\nabla^2(\phi_1^2 + \phi_2^2 + \phi_3^2) = 0,$$

hence we must have $\quad \dfrac{d\phi_1}{dx} = 0, \quad \dfrac{d\phi_1}{dy} = 0, \quad \dfrac{d\phi_1}{dz} = 0, \quad \dfrac{d\phi_2}{dx} = 0$, &c.;

$$\therefore\ \phi_1 = \text{const.}, \quad \phi_2 = \text{const.}, \quad \phi_3 = \text{const.}; \quad \therefore\ R = \text{const.}$$

5. On Clerk Maxwell's theory calculate the amount of the ether stress at the surface of the earth requisite to account for gravitation.

Here we have $R = g$; therefore $\frac{4}{3}\pi\rho\gamma a = g$, where ρ is the earth's mean density, and a is length of earth's radius.

Hence we get $\qquad E_2 = \dfrac{\rho g a}{6}.$

Now, adopting an inch as the unit of length, and observing that a cubic foot of water weighs $62\frac{1}{2}$ lbs., approximately; and adopting 6 for the mean density of the earth, and assuming $a = 4000$ miles, we get

$$E_2 = \frac{125 \times 4000 \times 5280 \times 12 \text{ tons}}{2 \times 12 \times 12 \times 12 \times 20 \times 112} = 4000 \text{ tons, nearly.}$$

Hence gravitation would require a vertical stress of about 4000 tons on the square inch, combined with a tension of equal amount in all horizontal directions. See Article on Attraction by Clerk Maxwell in the *Encyclopædia Britannica.*

I have given my calculation as above, since the value of the stress given by Clerk Maxwell is 37,000 tons per square inch.

CHAPTER IV.

Torsion of Prisms.

76. **Saint-Venant's Problem.**—We have seen in Art. 61 that in any problem of stress, if one system of displacements is found which satisfies all the conditions, it is the only solution of the problem. This method has been applied with great elegance by Saint-Venant to the treatment of the following general problem:—

If one end of a cylindrical substance be fixed, and if tangential shearing stress be applied over the other end, and if there be no stress on the lateral surface of the cylinder, to find the strain and stress at any point in its substance.

77. **Case of Pure Torsion.**—We commence by considering under what circumstances the strain is one of pure torsion, round an axis parallel to the edges of the cylinder.

Let us take this axis for the axis of z; then, by Art. 28, we have in this case

$$a = b = c = h = 0, \quad 2f = \tau x, \quad 2g = -\tau y.$$

Consequently, for an isotropic substance, we get

$$A = B = C = H = 0, \quad F = \mu\tau x, \quad G = -\mu\tau y. \qquad (1)$$

Again, for any plane parallel to the axis of z, making the angle a with the plane of zy, equations (3), Art. 45, give

$$T\cos\lambda = 0, \quad T\cos\mu = 0, \quad T\cos\nu = G\cos a + F\sin a.$$

From these we get $\cos\nu = 1$, *i.e.* $\nu = 0$, and we see that T is a tangential stress, parallel to the axis of z, whose amount is given by

$$T = F\sin a + G\cos a. \qquad (2)$$

Hence, since $T = 0$ over the lateral surface, we must have

$$F \sin a + G \cos a = 0, \qquad (3)$$

at all points on the boundary.

Consequently for a pure torsion we have $x \sin a - y \cos a = 0$ at every point on the boundary. Again, if ds be an element of the bounding curve in a plane perpendicular to the axis, we have

$$\sin a = \frac{dx}{ds}, \cos a = -\frac{dy}{ds};$$

hence round the boundary we must have

$$x\frac{dx}{ds} + y\frac{dy}{ds} = 0,$$

i. e.

$$x^2 + y^2 = \text{const.} \qquad (4)$$

This shows that for a pure torsion the cylinder must be a circular cylinder.

We infer that for a right circular cylinder acted on solely by tangential stress on its extreme faces, the strain is a pure torsion.

Also, we see that the strain cannot be a pure torsion for any other form of cylindrical or prismatic body.

78. **Stress Couple in this Case.**—We proceed to find an expression for the moment of the external stress couple, which applied to the free end of a right cylinder will produce a pure torsion of given amount. The stress on any element $dxdy$ of the end consists of the force $Fdxdy$, or $\mu\tau x dxdy$, acting parallel to the axis of y, along with $Gdxdy$, or $-\mu\tau y dxdy$, parallel to the axis of x.

Again, N the entire moment of these stresses about the axis of the cylinder is given by

$$N = \iint (Fx - Gy)\, dxdy = \mu\tau \iint (x^2 + y^2)\, dxdy.$$

But, if a be the radius of the cylinder, we have

$$\iint (x^2 + y^2)\, dxdy = \frac{\pi a^4}{2}.$$

Hence we get $$N = \frac{\pi}{2}\mu\tau a^4.$$

The quantity $\frac{N}{\tau}$ is called the torsional rigidity of the substance; denoting it by t, we have

$$t = \tfrac{1}{2}\mu\pi a^4. \tag{5}$$

This result was first obtained by Coulomb in his discussion of the theory of the torsion balance, and is usually called Coulomb's Law.

The preceding investigation holds good equally for a hollow cylinder. In that case the stress couple is given by

$$N = \frac{\pi}{2}\mu\tau(a^4 - b^4),$$

where a and b are the radii of the outer and inner cylinders, respectively.

Examples.

1. Show that the total stress at any point in the end of the cylinder varies directly as the distance of the point from the axis of the cylinder.

2. Compare the torsional rigidity of a solid cylindrical beam with that of a hollow cylinder of the same substance and the same external radius.

Let $b = ka$; and suppose M to be the masses of the solid, and M' that of the hollow cylinder, then

$$M' = (1 - k^2)M, \text{ and we get}$$

$$\frac{t'}{t} = (1 - k^4); \text{ therefore } \frac{t'}{M'} = \frac{t}{M}(1 + k^2).$$

Hence, proportionally to its mass, the resistance to torsion of a circular cylinder is increased by making it hollow.

79. **Case of Warping combined with Torsion.**—We have seen that the preceding solution fails when applied to non-circular cylinders. It was, however, shown by Saint-Venant that by a slight modification many other problems of cylindrical strain can be readily solved.

Let us suppose that the displacement w is no longer zero, but is some function of x and y, while the displacements u, v correspond to a pure torsion.

In this case the strain is represented by the equations

$$u = -\tau yz, \quad v = \tau xz, \quad w = f(x, y). \tag{6}$$

Again, since the displacement w is parallel to the axis of the cylinder, the sections parallel to the base of the cylinder are no longer plane sections after the strain. Accordingly the torsion represented by u and v is accompanied by what is styled *warping* of the cylinder.

Again, for the corresponding strains we have

$$a = b = c = h = 0; \; 2f = \frac{dw}{dy} + \tau x, \quad 2g = \frac{dw}{dx} - \tau y. \tag{7}$$

Hence we get

$$A = B = C = H = 0, \text{ and also}$$

$$F = \mu\left(\frac{dw}{dy} + \tau x\right), \quad G = \mu\left(\frac{dw}{dx} - \tau y\right). \tag{8}$$

Again, round the boundary we must have, as before,

$$F \sin a + G \cos a = 0,$$

$$\text{or } \left(\frac{dw}{dy} + \tau x\right)\frac{dx}{ds} - \left(\frac{dw}{dx} - \tau y\right)\frac{dy}{ds} = 0,$$

i. e.

$$\left(\frac{dw}{dy}\,dx - \frac{dw}{dx}\,dy\right) + \tau\,(xdx + ydy) = 0. \tag{9}$$

Hence, if the equation of the boundary be

$$\chi + \tfrac{1}{2}\tau\,(x^2 + y^2) = \text{const.}, \tag{10}$$

we must have

$$\frac{d\chi}{dx} = \frac{dw}{dy}, \quad \frac{d\chi}{dy} = -\frac{dw}{dx}. \tag{11}$$

These show, by Art. 24, that χ and w are *conjugate functions.*

Hence, if we can find any function χ which satisfies the equation

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} = 0, \tag{12}$$

then, if the expression

$$\chi + \tfrac{1}{2}\tau (x^2 + y^2) = \text{const.} \tag{13}$$

be taken as the equation of the boundary of a cylinder, the corresponding value of the warping function is *conjugate* to χ; and the strain due to a tangential stress action on the face of the cylinder, is represented by the foregoing expressions.

80. **Elliptical Cylinder.** — The elliptic cylinder furnishes the simplest application of the preceding; for if we assume

$$\chi = \frac{a}{2} (y^2 - x^2),$$

then we must have

$$w = - axy,$$

and the equation of the boundary becomes

$$(\tau + a) y^2 + (\tau - a) x^2 = \text{const.}$$

Hence, if the equation of the elliptic boundary be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we have

$$a = \tau \frac{a^2 - b^2}{a^2 + b^2},$$

and we see that the condition of strain of the cylinder is determined by the equations

$$u = - \tau yx, \quad v = \tau zx, \quad w = - \tau \frac{a^2 - b^2}{a^2 + b^2} xy. \tag{14}$$

The shearing stresses are in this case

$$F = \frac{2\mu\tau b^2}{a^2 + b^2} x, \quad G = - \frac{2\mu\tau a^2}{a^2 + b^2} y.$$

Again, if S be the whole shearing stress at any point, we have

$$S = \sqrt{F^2 + G^2} = 2\frac{\mu\tau a^2 b^2}{a^2 + b^2}\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}. \qquad (15)$$

It is obvious that S has its greatest value at an extremity of the minor axis of the ellipse.

Hence, if the end of the elliptic prism be subjected to an external couple, the extremities of the minor axis are the points at which rupture should first take place when the applied external couple is increased indefinitely.

Saint-Venant has shown that in all prisms the points nearest to the axis of the cylinder are those at which the shearing force tending to produce rupture is greatest.

81. **Whole Stress reducible to a Couple.**—It can be readily seen that whenever we can satisfy the conditions indicated in Art. 79, the entire stress action on the free end of the cylinder is reducible to a couple. This reduces to proving that the following equations

$$\iint F dx dy = 0, \quad \iint G dx dy = 0 \qquad (16)$$

hold good when the integrals are taken over the entire surface of the face of the cylinder.

For any area we have, by a well-known equation,

$$\iint\left\{\frac{d}{dy}(xF) + \frac{d}{dx}(xG)\right\} dx dy = \int x\left(F\frac{dx}{ds} - G\frac{dy}{ds}\right) ds, \quad (17)$$

where the single integral is taken round the boundary, and the double integral through all points in the area inclosed by the boundary.

Hence, by (3), we get

$$\iint\left\{\frac{d}{dy}(xF) + \frac{d}{dx}(xG)\right\} dx dy = 0$$

when taken as above, *i.e.*

$$\iint x\left(\frac{dF}{dy} + \frac{dG}{dx}\right) dx dy + \iint G dx dy = 0;$$

therefore $\qquad \iint G dx dy = 0, \;$ since $\dfrac{dF}{dy} + \dfrac{dG}{dx} = 0.$

In like manner we get

$$\iint F\,dxdy = 0.$$

Hence we see that the whole stress action on the face of the cylinder is equivalent to a couple. This couple is called the *torsion couple.* In the next Article we proceed to find a general expression for the magnitude of this couple.

82. **Torsion Couple.**—Denoting the stress couple by N we have

$$\begin{aligned} N &= \iint(Fx - Gy)\,dxdy \\ &= \mu\iint\left(x\frac{dw}{dy} - y\frac{dw}{dx}\right)dxdy + \mu\tau\iint(x^2+y^2)\,dxdy \\ &= \mu\iint\left(x\frac{d\chi}{dx} + y\frac{d\chi}{dy}\right)dxdy + \mu\tau\iint(x^2+y^2)\,dxdy. \end{aligned}$$

Again, if W be the potential energy of the strain we have, from the expression for V given in Art. 66,

$$\begin{aligned} W &= 2\mu l\iint(f^2+g^2)\,dxdy \\ &= \frac{\mu l}{2}\iint\left(\tau x + \frac{d\chi}{dx}\right)^2 + \left(\tau y + \frac{d\chi}{dy}\right)^2 dxdy, \end{aligned}$$

hence

$$W - N\frac{l\tau}{2} = \frac{\mu l}{2}\iint\left\{\frac{d\chi}{dx}\left(\frac{d\chi}{dx} + \tau x\right) + \frac{d\chi}{dy}\left(\frac{d\chi}{dy} + \tau y\right)\right\}dxdy.$$

It can now be shown that the last double integral vanishes when taken over the entire face of the cylinder.

For suppose the equation of the boundary, as before, to be

$$f(x, y) = \chi + \tfrac{1}{2}\tau(x^2+y^2) = \text{const.},$$

then, denoting the double integral by U, we have

$$\begin{aligned} U &= \iint\left(\frac{d\chi}{dx}\frac{df}{dx} + \frac{d\chi}{dy}\frac{df}{dy}\right)dxdy \\ &= \iint\left\{\frac{d}{dx}\left(f\frac{d\chi}{dx}\right) + \frac{d}{dy}\left(f\frac{d\chi}{dy}\right)\right\}dxdy, \text{ by (12).} \end{aligned}$$

Again, as in (17), the latter integral is equal to

$$\int f\left\{\frac{d\chi}{dx}\,dy - \frac{d\chi}{dy}\,dx\right\}$$

taken along the boundary.

Also, since f is constant along the boundary, we have

$$U = f\int\left(\frac{d\chi}{dx}\,dy - \frac{d\chi}{dy}\,dx\right)$$

$$= f\int\left(\frac{dw}{dx}\,dx + \frac{dw}{dy}\,dy\right).$$

Again, since the boundary is a closed curve, we have

$$\int\left(\frac{dw}{dx}\,dx + \frac{dw}{dy}\,dy\right) = 0,$$

and consequently $U = 0$;

hence we have

$$N = \frac{2W}{l\tau}. \tag{18}$$

Also, by (16), we see that the work done by the couple N is represented by

$$W = \tfrac{1}{2}l\tau N, \tag{19}$$

a result that might have been anticipated from physical considerations.

83. **Triangular Prism.**—We shall next determine in what case a triangular boundary can be found to satisfy equation (12).

Since the equation of the boundary must be of the third degree, we shall write it, by (13), in the form

$$\kappa r^3 \cos 3\theta + \tfrac{1}{2}\tau\,(x^2 + y^2) = \text{const.}$$

This may be written in the shape

$$x^3 - 3xy^2 + 3a\,(x^2 + y^2) + \beta = 0,$$

or

$$x^3 + 3ax^2 + \beta - 3y^2\,(x - a) = 0.$$

This is divisible by $x - a$ if $\beta = -4a^3$,

also $$(x^3 + 3a\,x^2 - 4a^3) = (x - a)(x + 2a)^2.$$

Again,

$$(x + 2a)^2 - 3y^2 = (x + 2a + y\sqrt{3})(x + 2a - y\sqrt{3}),$$

and the curve breaks up into the lines

$$x - a = 0, \quad x + y\sqrt{3} + 2a = 0, \quad x - y\sqrt{3} + 2a = 0.$$

It is easily seen that these form an equilateral triangle, whose central point is at the origin, and the length of each of whose sides is $2a\sqrt{3}$.

We thus see that a prism whose perpendicular section is an equilateral triangle satisfies the conditions of Art. 79.

Also since

$$r^3 \sin 3\theta = 3xy^2 - y^3,$$

we have

$$w = \kappa\,(3xy^2 - y^3) = \frac{\tau}{6a}\,(3xy^2 - y^3).$$

84. **General Case of Saint-Venant's Problem.**— From Art. 79 it follows that, for any given form of prism, the problem reduces to finding a function χ, of x, y, which shall satisfy the equation

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} = 0 \tag{20}$$

at all points in the body; and shall satisfy the equation

$$\chi + \frac{\tau}{2}(x^2 + y^2) = \text{const.} \tag{21}$$

at all points on the boundary.

It can be readily shown that the solution of this problem is unique; for if possible suppose that two functions χ, χ' both satisfy (20) and (21), and let $\chi - \chi' = u$.

Then the function u satisfies the equation $\frac{d^2u}{dx^2} + \frac{d^2u}{dx^2} = 0$, along with the equation $u = 0$ over the boundary.

Again, as in (17), we have

$$\iint\left\{\frac{d}{dx}\left(u\frac{du}{dx}\right)+\frac{d}{dy}\left(u\frac{du}{dy}\right)\right\}dxdy$$

$$=\int u\left(\frac{du}{dy}\frac{dx}{ds}-\frac{du}{dx}\frac{dy}{ds}\right)ds;$$

hence $$\iint\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right\}dxdy=0, \tag{22}$$

since $u = 0$ around the boundary, and $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ for all points within the boundary.

From this it follows that $\frac{du}{dx}=0$, $\frac{du}{dy}=0$ at all points within the area; hence u must be constant at all points; but we have $u = 0$ along the boundary; hence $u = 0$ everywhere, or $\chi=\chi'$.

The student is referred to Thompson and Tait's *Natural Philosophy*, vol. I., pp. 167–70, for a proof of Green's general theorem, of which the above is a particular case; as also for a proof that the problem admits in all cases of a real solution.

85. **Rectangular Prism.**—To find the strain and stress of a rectangular prism subjected to tangential forces on its end sections, as in Saint-Venant's problem.

Let the centre of the end face of the prism be taken as origin, and the coordinate axes parallel to the sides of the rectangle; also let $2a$ and $2b$ be the lengths of the sides: ~~then~~ we have $F = 0$ along one pair of sides, and $G = 0$ along the other pair.

Here the normal displacement w must satisfy the equation

$$\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}=0.$$

Also, when $x=\pm a$, we must have

$$\frac{dw}{dx}-\tau y=0,$$

and, when $$y=\pm b,\quad \frac{dw}{dy}+\tau x=0,$$

Now, let $w' = w - \tau xy$, and these conditions become

$$\frac{d^2w'}{dx^2} + \frac{d^2w'}{dy^2} = 0, \tag{23}$$

along with

$$\frac{dw'}{dx} = 0, \text{ when } x = \pm a, \tag{24}$$

and

$$\frac{dw'}{dy} + 2\tau x = 0, \text{ when } y = \pm b. \tag{25}$$

To solve this system of equations, let $w' = \phi \sin \alpha x$, where ϕ is a function of y, and α is a constant; then

$$\frac{d^2w'}{dx^2} = -\alpha^2\phi \sin \alpha x,$$

and equation (23) becomes

$$\frac{d^2\phi}{dy^2} = \alpha^2\phi.$$

This gives $$\phi = Ae^{\alpha y} + Be^{-\alpha y},$$

or $$w' = (Ae^{\alpha y} + Be^{-\alpha y}) \sin \alpha x.$$

Again, since the strain is plainly symmetrical with respect to the axis of x, w' must change its sign, but not its magnitude, when y is changed into $-y$; this gives $B = -A$, and we get

$$w' = A\,(e^{\alpha y} - e^{-\alpha y}) \sin \alpha x.$$

Again, since

$$\frac{dw'}{dx} = 0, \text{ when } x = \pm a,$$

we must have $\cos a\,\alpha = 0$, or

$$\alpha = \frac{(2n+1)}{a}\frac{\pi}{2}, \tag{26}$$

where n is any integer.

Hence, we see that (24) is satisfied if

$$w' = \sum_{n=0}^{n=\infty} A_n\,(e^{\alpha y} - e^{-\alpha y}) \sin \alpha x, \tag{27}$$

where α is determined by (26).

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Again, $$\frac{dw'}{dy} = \sum_{n=0}^{n=\infty} \alpha A_n (e^{\alpha y} + e^{-\alpha y}) \sin \alpha x.$$

In order that this should satisfy (25) we must have

$$-2\tau x = \sum_{n=0}^{n=\infty} \alpha A_n (e^{\alpha b} + e^{-\alpha b}) \sin \alpha x$$

$$= \sum_{n=0}^{n=\infty} \alpha B_n \sin \alpha x, \text{ suppose.}$$

To determine the coefficients, multiply both sides by $\sin \alpha x\, dx$, and integrate between the limits 0 and a; then since when

$$\alpha = \frac{2n+1}{a}\frac{\pi}{2}, \text{ and } \alpha' = \frac{2n'+1}{a}\frac{\pi}{2},$$

we have

$$\int_0^a \sin \alpha x \sin \alpha' x\, dx = 0;$$

we get

$$-2\tau \int_0^a x \sin \alpha x\, dx = \alpha B_n \int_0^a \sin^2 \alpha x\, dx;$$

but

$$\int_0^a x \sin \alpha x\, dx = \frac{(-1)^n}{\alpha^2}, \text{ and } \int_0^a \sin^2 \alpha x\, dx = \frac{a}{2}.$$

Accordingly

$$B_n = (-1)^{n+1} \frac{4\tau}{\alpha^3 a}.$$

Hence (27) becomes

$$w' = \frac{32\tau a^2}{\pi^3} \sum_{n=0}^{n=\infty} \frac{(-1)^{n+1}}{(2n+1)^3} \frac{e^{\alpha y} - e^{-\alpha y}}{e^{\alpha b} + e^{-\alpha b}} \sin \alpha x.$$

Consequently,

$$w = \tau xy - \frac{32\tau a^2}{\pi^3} \sum_{n=0}^{n=\infty} \frac{(-1)^n}{(2n+1)^3} \frac{(e^{\alpha y} - e^{-\alpha y})}{e^{\alpha b} + e^{-\alpha b}} \sin \alpha x, \quad (28)$$

where $\alpha = \frac{2n+1}{a}\frac{\pi}{2}$, and the summation extends through all values of n from 0 to ∞.

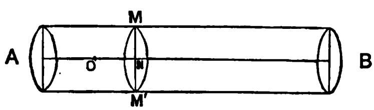
Also, by Art. 84, we see that this is the only solution that the problem admits of.

CHAPTER V.

ELASTIC BEAMS.

86. **Loaded Beams, Bending Moment.**—If we conceive an elastic body, which is in equilibrium under the action of any forces, to be divided arbitrarily into two parts, then the system of external forces that act on *either* part of the body must be in equilibrium with the system of external stresses that act over the surface by which the two parts are separated. We shall apply this elementary principle to the case of a uniform beam, which is supported in a horizontal position, and is such that when unstrained its axis AB is horizontal. We shall also suppose that the external forces all act in the *vertical* direction, and are either uniformly distributed over the whole beam or portions of the beam, or else act at given points on its surface; including among the latter the pressures arising from its supports.

Suppose any vertical section drawn perpendicular to the axis AB, and let MM' be the vertical line drawn through the point in which the section intersects AB; let any point O on the axis be taken as origin, and let $ON = x$, $OB = b$.



Then any vertical pressure P_1 is equivalent to an equal pressure acting along MM' together with a couple; consequently any number of vertical pressures are equivalent to a single force acting along MM' together with a single couple. We shall suppose that the external forces are symmetrically distributed so that the resultant couple may be regarded as acting in the vertical plane MBM' that passes through the axis. Let us denote by P_1, P_2, &c., the pressures that act at *isolated points*; such pressures being regarded as *positive* when they act in the

direction of gravity; then the part of the resultant along MM' arising from this system is represented by ΣP.

Again, if the beam be uniformly loaded the corresponding load on BN is represented by $w(b-x)$, where w is the load per unit of length. Combining these two results, the entire force R along MM' is given by the equation

$$R = \Sigma P + w(b-x). \qquad (1)$$

Again, the resultant couple, since it tends to bend or rupture the beam, is called the *bending moment* at the section. We shall denote by $\mathbf{M}_x$ the bending moment at the distance x from the origin, and we shall regard it as *positive* when it tends to bend the axis *downwards*.

87. **Expression for Bending Moment.**—For any external pressures, such as those considered in the previous Article, the bending moment at any point x is a quadratic function in x. This is easily seen: for the bending moment corresponding to any pressure P_1, acting at the distance a_1 from O, is $P_1(a_1 - x)$; also the bending moment from the uniform load w is plainly $\frac{1}{2}w(b-x)^2$; accordingly we have

$$\mathbf{M}_x = \Sigma(P_1a_1) - x\Sigma P_1 + \tfrac{1}{2}w(b-x)^2 \qquad (2)$$

$$= A - Bx + \tfrac{1}{2}wx^2, \qquad (3)$$

where $\quad A = \Sigma(P_1a_1) + \frac{1}{2}wb^2, \quad B = \Sigma P_1 + wb.$

It is evident that A is the bending moment at the origin.

The bending moments for the different points of a beam can be represented by a diagram, for if the ordinate y be drawn proportional to the bending moment $\mathbf{M}_x$, then the parabola

$$y = A - Bx + \tfrac{1}{2}wx^2, \qquad (4)$$

when drawn to scale, is called the diagram of bending moments, and gives the bending moment at each point. It is plain that the diagram in general consists of a series of parabolic arcs, each intersecting the next in a line in which one of the isolated vertical forces P_1, P_2, &c. acts. Since we have here supposed the *whole* beam to be uniformly loaded, the parabolæ, having the same *latus rectum*, will be all equal,

but differ in position. The foregoing method can be readily extended to the case in which the value of w is different for different portions of the beam. For full details, with figures and application to moving loads, the student is referred to Alexander and Thomson's *Elementary Applied Mechanics*, Part 2.

Again, it is obvious from (2) that we have

$$R = -\frac{d\mathbf{M}_x}{dx}, \qquad (5)$$

a result that will be generalized in Article 90.

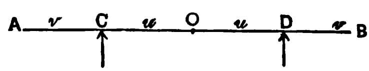
From (5) we see that $\mathbf{M}_x$ is a maximum or a minimum at the points at which the corresponding force R vanishes.

It is easily seen that in general the bending moment is more effective in breaking a beam than the corresponding shearing force R; hence for any uniform beam of homogeneous material, if we suppose the load to be uniformly increased so as to rupture the beam, the beam will break at the section at which the bending moment is a maximum regardless of sign.

This principle enables us in many cases to find the greatest uniform load consistent with the safety of the beam; and it also enables us to determine in certain cases the best position of props for supporting a given load; *i.e.* their position in order to support the greatest uniform load without injury to the elasticity of the beam.

88. **Position of Two Props for Maximum Load.**— For example, to find where two props should be placed in supporting a *uniformly* loaded horizontal beam.

Let AB represent the axis of the beam, and C, D the positions of the supporting props. It is easily seen that the beam will support



the greatest load, without rupture, when the bending moments at C and D are equal, and when they are also each equal, but *opposite in sign*, to the maximum bending moment between C and D; for if these moments are not all equal, one of the props can be moved so as to diminish the greatest. In order that the bending moments at C and D

should be equal the props must be equidistant from the extremities of the beam.

Again, let $OC = OD = u$, $AC = BD = v$. Since the bending moments at C and D are equal, it is plain that for points between C and D the bending moment is greatest (regardless of sign) at the middle point O.

If the bending moment at O be denoted by $\mathbf{M}_0$, we have

$$\mathbf{M}_0 = \tfrac{1}{2}w(u+v)^2 - w(u+v)u = \tfrac{1}{2}w(v^2 - u^2), \qquad (6)$$

where w is the load per unit of length, as before.

Also $$\mathbf{M}_c = \tfrac{1}{2}wv^2.$$

Accordingly, for the position of greatest strength, we have $$\mathbf{M}_c = -\mathbf{M}_0, \quad \text{or} \quad u^2 = 2v^2, \ i.e.\ u = v\sqrt{2}. \qquad (7)$$

This determines the best position of the props in order to support a uniform load.

EXAMPLES.

1. A uniform beam AB is supported at each end; prove that the bending moment at any point P in its length is $\frac{1}{2}wAP \,.\, PB$.

2. Compare the strength for resisting rupture of a uniform beam supported at each end, with that when the props are placed as in Art. 88.

Ans. The latter is stronger in the ratio $3 + 2\sqrt{2} : 1$.

3. A uniform horizontal beam which is to be equally loaded at all points of its length, is supported at one end and at some other point; find where the second support should be placed in order that the greatest possible load may be placed on the beam without breaking it, and show that it will divide the beam in the ratio 1 to $\sqrt{2} - 1$. [*Camb. Math. Trip.*]

4. Compare the strength of the beam for resisting rupture in the previous example with that of the same beam when the props are situate as in Art. 88.

Ans. The beam will only bear half the load without rupture that it would when the props are placed as in Art. 88.

5. A horizontal beam, which is supported at both ends, and whose weight is neglected, is loaded with any number of isolated weights; if the bending moments be equal at the points of application of any pair of contiguous weights, they have the same value at all intermediate points.

6. A horizontal beam, whose weight is neglected, is supported at its extremities A, B, and is traversed by a moving load W distributed equally over a segment PQ of constant length; show that the bending moment at any point X of the beam is greatest when X divides PQ in the same ratio as that in which it divides AB; and show that this maximum bending moment is

$$\frac{W}{AB^2} AX \,.\, XB \left(AB - \tfrac{1}{2}PQ\right). \qquad \text{[TOWNSEND.]}$$

89. **Shearing Stress and Moment of Resistance to Rupture at any Section of the Beam.**—We have seen that, in the case here discussed, the whole system of external forces that act on any portion MBM' of the beam are reducible to a single vertical force R_x together with a single couple $\mathbf{M}_x$, which acts in the vertical plane MBM'. We now proceed to consider the equilibrating forces arising from the elastic stresses that act on the section MM'. The stress on any element of area in this section may be resolved into a stress normal to the section together with a tangential stress, that acts in the plane of the section. This system of tangential stresses is, in general, equivalent to a single stress along MM' together with a couple which acts in the plane of the section; but this couple must, in the case here considered, be evanescent, since there is no equilibrating couple arising from the external load. Also, for equilibrium, the resultant vertical stress along MM' must equilibrate the force R_x. This vertical stress, taken in the opposite direction, is usually called the *shear* in the plane section, and if we represent this shear by S_x we have

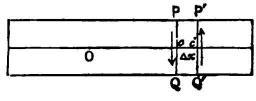
$$S_x = R_x = -\frac{d\mathbf{M}_x}{dx}. \tag{8}$$

Again, the system of normal stresses likewise must equilibrate the bending moment $\mathbf{M}_x$. Accordingly the system of normal stresses over the section must be reducible to a single couple.

If we denote this couple, taken in the opposite direction, by μ_x, we have $\mu_x = \mathbf{M}_x$.

This couple tends to resist the rupture of the beam arising from the corresponding bending moment, and is called the moment of resistance to rupture.

90. **General Proof of Equation** $S = -\dfrac{d\mathbf{M}}{dx}$.—In the general case of a beam subjected to *any vertical* load let us consider the equilibrium of an indefinitely small portion $PQQ'P'$ of the beam which lies between two parallel sections, each drawn perpendicular to the axis of the beam. Then



since no isolated external force acts at any point on this portion of the beam, the moment of the applied forces, acting on the surface and throughout the mass of $PQQ'P'$, round a horizontal axis in one of its vertical sections, is an infinitely *small quantity of the second order*.

Let $\mathbf{M}$ be the bending moment for the section PQ, $\mathbf{M} + \Delta\mathbf{M}$ that for section $P'Q'$. Also let S and $S + \Delta S$ be the corresponding shears. Then, since both $\mathbf{M}$ and S vary continuously in the neighbourhood of PQ, both $\Delta\mathbf{M}$ and ΔS become indefinitely small along with Δx.

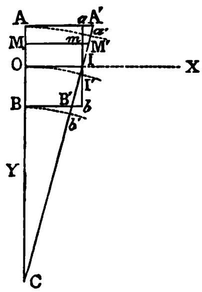
Again, observing that the bending moments act in opposite directions at the opposite faces of the lamina, they are equivalent to the single moment $\Delta\mathbf{M}$.

In like manner the shearing stresses S and $S + \Delta S$, acting in opposite directions, are equivalent to the couple $S\Delta x$, neglecting the force ΔS in comparison with S. For equilibrium these couples must be equal and opposite, and accordingly we have

$$S = -\frac{\Delta\mathbf{M}}{\Delta x} = -\frac{d\mathbf{M}}{dx}, \tag{9}$$

in the limit.

91. **Deflection. Neutral Surface.**—When a beam is acted on by any load, one of its surfaces becomes convex and the other concave, and the fibres are at one side *extended* and at the other *compressed*. The surface which separates these two portions of the beam, and which is neither extended or compressed, is called the *neutral surface* of the beam. We shall assume* that the strain is such that the molecules in any section that is perpendicular to the axis of the beam continue after flexion to lie in a plane.



Again, in considering the strain, we may, as in Art. 5, suppose the section whose trace is AB to be brought back to its original position, then any very near parallel section, whose trace is ab before

* The validity of the assumptions made in this Article will be considered in Art. 107.

the strain, will after the strain occupy a new position $a'b'$, as in the figure, normal to the arcs Aa' and Bb'. Let AB and $a'b'$ when produced meet in O, then O will be the centre of curvature of the arcs, which may be regarded as small arcs of circles, having O as their common centre. Also as $A'a'$ and $B'b'$ are very small of the *second order*, we may regard $A'B'$ as the new position of the section ab. Through I the point of intersection of ab and $A'B'$ draw OI parallel to AA', then all the fibres above the horizontal plane drawn through OI are extended, and those below are contracted. The horizontal plane through OI is called the neutral surface of the beam, since all the fibres in this plane are neither elongated nor contracted.

We shall assume that the corresponding stresses for the different fibres of the beam follow Hooke's law; thus if s_x and f_x be the strain and the corresponding stress for the fibre Mm, we have

$$f_x = Es_x\,d\mathrm{s} = Ed\mathrm{s}\,\frac{mM'}{Mm},$$

where E denotes the coefficient of elasticity, which we suppose the same for all fibres of the beam, and $d\mathrm{s}$ is the section of the fibre.

Again, by similar triangles we have

$$s_x = \frac{mM'}{Mm} = \frac{M'I}{IC}.$$

Here $M'I$ is the distance of the fibre Mm from the neutral surface: if we denote this distance by v, and the radius of curvature by ρ, we have

$$s_x = \frac{v}{\rho}, \text{ and } f_x = Ed\mathrm{s}\,\frac{v}{\rho}. \qquad (10)$$

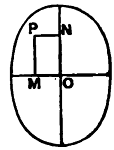
Accordingly, since by hypothesis $\Sigma f_x = 0$, we have

$$\Sigma v\,d\mathrm{s} = 0. \qquad (11)$$

This shows that the centre of gravity of the area of the section AB must lie in the *neutral surface*; and from the symmetry of the section, it may be taken at the point O.

The fibre which passes through the centres of gravity of all these parallel sections is called the *mean or neutral fibre* of the beam. It is also called the *neutral axis* of the beam.

92. **Expression for Moment of Resistance to Bending.**—Let us suppose the section perpendicular to the axis at O to be represented as in the figure: let P denote the point in which this plane intersects any fibre, and let



$$PM = v, \quad PN = u;$$

then the system of forces f_x, or $\frac{E}{\rho}\, vds$, give rise to the couples $\Sigma \frac{E}{\rho} v^2\, ds$ and $\Sigma \frac{E}{\rho} uvds$,

of which the former acts in the vertical plane drawn through ON, and the latter is parallel to the neutral plane. Again, we must have

$$\Sigma \frac{E}{\rho} uvds = 0, \tag{12}$$

since, by hypothesis, there is no corresponding couple arising from the load.

Also the couple $\frac{E}{\rho}\, \Sigma v^2 ds$, *when reversed,* must be in equilibrium with the bending moment, and is in fact the couple that resists the rupture of the beam at the section.

Moreover, the expression $\Sigma v^2 ds$ is the moment of inertia of the area of the section relative to the line OM. Denoting this moment of inertia by I, we have

$$\mu = \mathbf{M} = \frac{EI}{\rho}. \tag{13}$$

Again the equation $\Sigma uvds = 0$ shows that the preceding theory does not hold good when the vertical and horizontal lines OM and ON are not the *principal axes* for the section.

Again, if T be the stress, per unit of area, for any fibre Mm, we have

$$T = \frac{f_x}{ds} = Es_x = E\frac{v}{\rho}. \tag{14}$$

If ρ be eliminated between the two preceding equations we get

$$T = \frac{\mathbf{M}v}{I}. \tag{15}$$

We hence see that the stress for any fibre is proportional to its distance from the neutral surface. Accordingly the extreme fibres, *i.e.* those furthest from the neutral surface undergo the greatest tension or compression. Consequently the load should be so regulated that the greatest value of T should not be greater than the limits of elasticity for the beam admit of. Thus if T_1 denote the *proof stress* or the greatest value of T consistent with the elastic safety of the beam, and y_0 the distance of the extreme fibre, we get for the maximum bending moment consistent with safety,

$$\mathbf{M} = \frac{T_1 I}{y_0}, \text{ or } T_1 = \frac{\mathbf{M}y_0}{I}. \tag{16}$$

This may also be written in the form

$$\frac{Ey_0}{\rho}. \tag{17}$$

93. **Values of I for different Sections.**—For the convenience of reference the values of I in some of the cases commonly considered are here given. (See *Integral Calculus*, Chapter X.).

1. Rectangular section of breadth b and depth d. $I = \frac{1}{12} bd^3$.
2. Beam of circular section r. $I = \frac{1}{4}\pi r^4$.
3. Circular ring of small uniform thickness t, and radius r. $I = \pi r^3 t$, *approx.*
4. Isosceles triangle of base b, and perpendicular depth d. $I = \frac{1}{12} bd\left(\frac{1}{4}b^3 + \frac{1}{3}d^2\right)$.
5. Ellipse, with axis minor horizontal. $I = \frac{\pi}{4} ba^3$.
6. Ellipse, with axis major horizontal. $I = \frac{\pi}{4} ab^3$.

94. **Strongest Form of Beam.**—The best form that can be given to a beam, for strength, is that which renders it equally liable to rupture at every point, so that when the load is increased to its utmost limit the beam is on the point of rupture at all its points. The strongest beam of a given material is evidently that which can be constructed of the same strength with the least amount of material. Thus for the beam of strongest form the value of T_1 must be the same for all sections of the beam, and accordingly we must have

$$\frac{My_0}{I} = \text{const.}$$

along the beam. The greatest value the constant can have is determined in each case from the nature of the material, and by aid of tables on the strength of different substances.

If we suppose the section of the beam to be rectangular, of constant breadth b, but of variable depth d; then we have

$$y_0 = \tfrac{1}{2}d, \quad I = \tfrac{1}{12}bd^3;$$

consequently for a beam of uniform strength $\frac{M}{d^2}$ must be constant. Hence in this case the depth of each section should vary as the square root of the corresponding bending moment.

If the depth of a rectangular beam be constant, while its breadth is variable, then we should have

$$\frac{M}{b} = \text{const.},$$

and the breadth varies as the bending moment.

Again, if the section be circular we have

$$y_0 = r, \quad I = \tfrac{1}{4}\pi r^4;$$

and consequently $\frac{M}{r^3}$ must be constant; accordingly the radius of the section should vary as the cube root of the bending moment. Also, we see from (17) that whenever y_0 is constant, the curve formed by the neutral fibre, for a beam of uniform strength, is an arc of a circle.

95. Differential Equation of Neutral Fibre.—If y be the ordinate of any curve we have

$$\frac{1}{\rho} = \frac{\ddot{y}}{(1 + \dot{y}^2)^{\frac{3}{2}}}.$$

Substituting in (13) we get

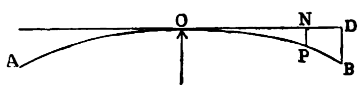
$$\frac{\ddot{y}}{(1 + \dot{y}^2)^{\frac{3}{2}}} = \frac{\mathbf{M}}{EI}.$$

If now we suppose that the deflection of the mean fibre is so small that $\dot{y}$ or $\frac{dy}{dx}$ may be neglected at all points, we have for the differential equation of the curve formed by the mean fibre,

$$\ddot{y} = \frac{\mathbf{M}_x}{EI}. \tag{18}$$

We proceed to apply this equation to find the equation of the mean fibre in a few simple cases.

96. Beam Supported at its Centre.—Suppose AB to represent the mean fibre of a uniform and uniformly loaded beam, which is supported at its centre O.



Take O as the origin, and OD as the axis of x; and also let x, y be the coordinates of any point P on the curve; then if, as before, w denote the load per unit of length, and $OB = a$, we obviously have

$$\mathbf{M}_x = \frac{w}{2}(a - x)^2;$$

$$\therefore \quad EI\ddot{y} = \tfrac{1}{2}w(a - x)^2; \tag{19}$$

hence $$EI\dot{y} = -\tfrac{1}{6}w(a - x)^3 + C.$$

Again, as $\dot{y} = 0$ at the origin, we get

$$C = \tfrac{1}{6}wa^3;$$

$$\therefore \quad EI\dot{y} = \tfrac{1}{6}w(x^3 - 3ax^2 + 3a^2x);$$

hence, since $y = 0$ when $x = 0$, we have

$$EIy = \tfrac{1}{6}w\left(\tfrac{1}{4}x^4 - ax^3 + \tfrac{3}{2}a^2x^2\right) = \frac{wx^2}{24}(x^2 - 4ax + 6a^2). \qquad (20)$$

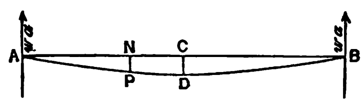
This gives the equation of the neutral axis.

Again, if $\delta = BD$, the *droop* at either extremity of the beam, we have

$$\delta = \tfrac{1}{8}w\frac{a^4}{EI}. \qquad (21)$$

Also, since the tangent at O is horizontal, the preceding results hold good for a uniformly loaded beam which is *solidly imbedded in masonry* at O, in a horizontal position.

97. **Beam supported at its Extremities.**—For a uniformly loaded beam supported at its extremities, let AB represent the unstrained position of the neutral fibre, C its middle point, and let ADB be its position when subjected to a uniform load. Also let $AC = a$, and w the load per unit of length; then the pressures on the supports A, B, are each equal to wa.



Hence for $\mathbf{M}_x$, the bending moment at P, we have

$$\mathbf{M}_x = \tfrac{1}{2}wx^2 - wax.$$

Consequently $\qquad EI\ddot{y} = \tfrac{1}{2}wx^2 - wax;$

$$\therefore \quad EI\dot{y} = \tfrac{1}{6}wx^3 - \tfrac{1}{2}wax^2 + C, \qquad (22)$$

where C is an arbitrary constant.

Integrating again, we have, since $y = 0$ when $x = 0$,

$$EIy = \tfrac{1}{24}wx^4 - \tfrac{1}{6}wax^3 + Cx.$$

Also since $y = 0$ when $x = 2a$, we get

$$C = \tfrac{1}{3}wa^3. \qquad (23)$$

Hence the equation of the mean fibre is

$$EIy = \frac{wx}{24}(x^3 - 4ax^2 + 8a^3) = \frac{wx(2a - x)}{24}(4a^2 + 2ax - x^2). \qquad (24)$$

Next, suppose the whole load on the beam to be a single weight W applied at its middle point; then the pressures at A and B are each $\tfrac{1}{2}W$. Consequently the bending moment

at P is $-\frac{1}{2}Wx$, and the differential equation of the mean fibre is

$$EI\ddot{y} = -\tfrac{1}{2}Wx;$$

$$\therefore\quad EI\dot{y} = -\tfrac{1}{4}Wx^2 + C.$$

Again, since $\dot{y} = 0$ when $x = a$, we have

$$C = \tfrac{1}{4}Wa^2;$$

$$\therefore\quad EI\dot{y} = \tfrac{1}{4}W(a^2 - x^2). \tag{25}$$

Hence, at points between A and C we have

$$EIy = \tfrac{1}{4}Wx\left(a^2 - \frac{x^2}{3}\right). \tag{26}$$

98. **Droop and Slope of Beam, Stiffness.**—For any beam the depth of its lowest point is called its droop, and the angle the tangent at either extremity makes with the horizontal line is called the slope at that extremity.

Thus, if δ be the droop, we have by (24), for a uniformly loaded beam,

$$\delta = \tfrac{5}{24}\frac{wa^4}{EI} = \tfrac{5}{48}\frac{Wa^3}{EI}. \tag{27}$$

Also, if α be the slope at A, we get by (22),

$$C = EI\tan\alpha;$$

accordingly, by (23), $\tan\alpha = \dfrac{Wa^2}{6EI}$; (28)

also, the equation $\delta = \frac{5}{8}\, a\tan\alpha$ connects the droop with the slope.

If ρ be the radius of curvature at D, regardless of its sign, we have

$$\frac{EI}{\rho} = \tfrac{1}{2}wa^2.$$

Hence $$\rho\delta = \tfrac{5}{12}a^2. \tag{29}$$

Again, in the case of a load W placed at the middle point of the beam, we have

$$\delta = \tfrac{1}{6}\frac{Wa^3}{EI},\quad \tan\alpha = \tfrac{1}{4}\frac{Wa^2}{EI}.$$

Comparing these results we see that the droop of a uniformly loaded beam is $\frac{5}{8}$ of its droop if the whole load were concentrated at its centre; also that the slope in the former case is $\frac{2}{3}$ of the slope in the latter.

The stiffness of a beam is measured by the ratio of its droop at the middle point to its length.

Again, if a uniformly loaded beam be subjected to a vertical pressure P at its middle point, its total droop at that point is obtained by combining the preceding results, we accordingly have in this case

$$EI\delta = \frac{a^3}{6}\left(\tfrac{5}{8}W + P\right).$$

This result still holds good if P act *upwards*, in which case we have

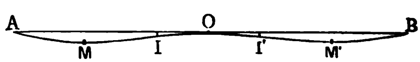
$$EI\delta = \frac{a^3}{6}\left(\tfrac{5}{8}W - P\right). \qquad (30)$$

Let us now suppose the upward pressure such that $\delta = 0$, we get in that case

$$P = \tfrac{5}{8}W. \qquad (31)$$

99. **Beam supported at Extremities and at Middle Point.**—If a uniformly loaded beam be supported by props at its extremities, and also at its middle point, the three extremities of the props being situated in the same horizontal line, we see, by (31), that the middle prop will support $\frac{5}{8}$ of the entire load; accordingly the pressures on the extreme props are each $\frac{3}{16}$ of the load.

It is easy to obtain the equation of the neutral fibre in this case. For let O be the middle point of the beam AB; then



taking O as the origin, and OB as the axis of x, we get for $\mathbf{M}_x$, the bending moment at the distance x,

$$\mathbf{M}_x = \tfrac{1}{2}w(a-x)^2 - \tfrac{3}{8}wa(a-x),$$

or

$$EI\ddot{y} = \frac{w}{8}(a^2 - 5ax + 4x^2). \qquad (32)$$

Hence, observing that $\dot{y} = 0$ when $x = 0$, we have

$$EI\dot{y} = \tfrac{1}{8}w\,(a^2x - \tfrac{5}{2}ax^2 + \tfrac{4}{3}x^3)\,;$$

accordingly, since the curve passes through the origin,

$$EIy = \tfrac{1}{8}w\,(\tfrac{1}{2}a^2x^2 - \tfrac{5}{6}ax^3 + \tfrac{1}{3}x^4)$$

$$= \tfrac{1}{48}wx^2\,(3a^2 - 5ax + 2x^2)$$

$$= \tfrac{1}{48}wx^2\,(a - x)\,(3a - 2x). \qquad (33)$$

This gives the equation of the mean fibre.

Again, if $OI = OI' = \tfrac{1}{4}OA$, we see from (32) that the bending moment vanishes at the points I and I'; these points are *points of inflexion* on the mean fibre; they are sometimes called the *hinges of the beam*, since, as there is no moment to bend the beam, hinges may be introduced into the beam at these points without altering the conditions of the problem.

Again, the bending moment is a maximum when $x = \tfrac{5}{8}a$.

Accordingly, if $OM = OM' = \tfrac{5}{8}OA$, we get the points at which the upward bending moment is a maximum, *i.e.* the points between A and I, and between B and I', at which the beam is most liable to rupture.

It is important to consider whether the beam would first rupture at O, or at M and M', when we suppose the uniform load to be increased sufficiently. This depends on whether the downward bending moment at O or the upward bending moment at M is the greater.

Now from (32), the former bending moment is $\tfrac{1}{8}wa^2$, while the latter is $\tfrac{9}{128}wa^2$; consequently the middle point is the weakest point of the beam.

100. **Strongest Form of Beam.**—We shall next consider the best form for strength in the cases treated of in the preceding Articles.

First, suppose the section rectangular and of *constant breadth*, then denoting by y the depth at the point x, we get,

by Art. 96, for the shape of the upper and lower surfaces of a beam supported at its middle point,

$$y^2 = k^2 (a - x)^2,$$

or

$$y = k (a - x),$$

where k is a constant, showing that the surfaces are planes in this case.

In the same case, if the *depth* is constant, and if we denote the breadth by z, the form of the beam is given by the parabola

$$kz = (a - x)^2,$$

where k is a constant.

Again, in Art. 97, we have $\mathbf{M} = \frac{1}{2}wx (2a - x)$, consequently for a beam of constant breadth supported at its extremities, and having a rectangular cross section, the form of maximum strength is given by the equation

$$y^2 = kx (2a - x). \tag{34}$$

This represents an ellipse whose centre is at the middle point of the beam.

If the depth be constant we get a parabola as in the previous case. It should be observed that whenever *the depth is constant*, since the breadth varies as the bending moment, the diagram of bending moment furnishes us with the shape of the cross section of the beam of maximum strength.

Examples.

1. Compare the droops of two rectangular beams each under the same load, according as the greater or lesser side of the rectangle is vertical.

2. A uniform elastic beam rests, in non-limiting equilibrium, with one end on the ground and the other against a vertical wall, the vertical plane through the beam being at right angles to the wall, investigate the form of the mean fibre of the beam.

If α be the inclination of the beam, we may resolve the weight of the beam into two components, one perpendicular and the other parallel to the beam. The deflection of the beam is caused solely by the perpendicular component, and accordingly the problem is a case of that considered in Art. 97.

3. If a beam be supported at its middle point show that the droop of either end is $\frac{8}{5}$ of the droop of the middle point of the same beam when it is supported at its extremities.

4. The droop of the middle point of a uniform beam which is supported at its ends, is increased in the ratio of 5 : 13 by laying a mass equal to its own mass at its middle point.

5. If the beam be supported at its middle point, prove that the droop of each end is increased in the ratio 3 : 11 by hanging to each end a weight equal to half the weight of the beam.

6. If I and I' be two consecutive points of inflexion for any portion of a uniformly loaded beam, prove that for any point P between I and I' the bending moment varies as the rectangle under IP and PI'.

7. A uniform elastic rod rests on three supports, one under its middle point, the others at its ends. Find how much the middle support must be lower than the end supports in order that the pressures on the three supports shall be equal. (*London Univ.*, 1881.)

8. A uniform girder rests with one extremity on a prop, while the other is firmly built into a wall; find the equation of the neutral fibre.

Ans. $EIy = \frac{w}{48} x (l - x)^2 (l + 2x)$, where l is the length of the beam.

9. Compare the maximum deflections of a uniform wire in a horizontal position when clamped at one end, and when resting on two supports under its ends.

10. A beam is supported at its ends and loaded, with a weight W, at a point P. Show that the depression of the point P is $\frac{Wa^2b^2}{3EI(a+b)}$, where a, b, are the distances from P to the points of support, the weight of the beam being neglected.

11. A hollow circular pillar of uniform thickness being subjected to the straining action of any external system of transverse forces, show that its stiffness to resist change of form by bending is to that of a solid pillar of the same length, under the action of the same system of forces, as the sum is to the difference of the squares of its outer and inner radii. (*London Univ. Exam.*, 1878.)

12. Find the form of the rectangular girder of maximum strength cut out of a cylinder.

The strength is a maximum when bd^2 is a maximum. But we have $b^2 + d^2 = 4r^2$, where r is the radius of the cylinder; $\therefore d = b\sqrt{2}$.

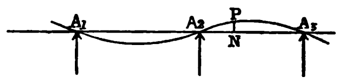
13. Compare the droop δ of a uniform rectangular beam, subject to a uniform load, and supported at its extremities, with that δ_1 of a rectangular beam of uniform strength, and uniform depth, similarly supported, and subject to the same uniform load. The beams are supposed to be of equal strength, and of equal depth and length: and the weight of each *beam is neglected in comparison* with the load that it supports.

The curvature of the beam of maximum strength, since its depth is constant, is uniform throughout by (17); and therefore the mean fibre in this case is an arc of a circle. Again, it is readily seen, since the strength of the beam is the same as that of the uniform rectangular beam, that the radius of this circle is equal to ρ, the radius of curvature at the lowest point of the uniform beam. Again, from an elementary property of the circle, we have $\rho\delta_1 = \frac{1}{2}a^2$, *approx.*, where $2a$ is the length of the beam.

Hence from equation (29) we get $\delta_1 = \frac{6}{5}\delta$.

101. Equation* of Three Moments.—Unless the contrary be stated we shall, in all the following problems concerning beams, assume that the *points of support are all situated in a common horizontal line.*

If a uniform and uniformly loaded horizontal beam rest on a number of supports, the bending moments at any three consecutive supports are connected by a remarkable relation, which we now proceed to establish.



Let A_1, A_2, A_3, be the points in the neutral fibre that correspond to three consecutive supports; and let the corresponding bending moments be denoted by $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$, respectively.

Then, taking A_2 as the origin, and A_2A_3 as the axis of x, that of y being vertically downwards, the differential equation of the curve A_2PA_3 may be written by (18),

$$EI\ddot{y} = \mathbf{M}_2 + Bx + \tfrac{1}{2}w_2x^2, \qquad (35)$$

where B is undetermined, and w_2 is the load on A_2A_3 per unit of length.

Hence $\quad EI\dot{y} = \mathbf{M}_2x + \tfrac{1}{2}Bx^2 + \tfrac{1}{6}w_2x^3 + C.$

Again, if a_2 be the angle that the tangent at A_2 makes with the axis of x, we have

$$C = EI \tan a_2;$$

$$\therefore \quad EI(\dot{y} - \tan a_2) = \mathbf{M}_2x + \tfrac{1}{2}Bx^2 + \tfrac{1}{6}w_2x^3; \qquad (36)$$

$$\therefore \quad EI(y - x\tan a_2) = \tfrac{1}{2}\mathbf{M}_2x^2 + \tfrac{1}{6}Bx^3 + \tfrac{1}{24}w_2x^4. \qquad (37)$$

Now let $\quad A_1A_2 = a, \quad A_2A_3 = b,$

then, since when $x = b$ we have $y = 0$, we get

$$-EI\tan a_2 = \tfrac{1}{2}\mathbf{M}_2b + \tfrac{1}{6}Bb^2 + \tfrac{1}{24}w_2b^3. \qquad (38)$$

---

* *The theorem of three moments* is usually ascribed to Clapeyron; it is originally due however to M. Bertot, by whom it was published in 1855: see Collignon, *Cours de Mécanique*, tome 1, p. 243.

Also from (35),

$$\mathbf{M}_3 = \mathbf{M}_2 + Bb + \tfrac{1}{2}w_2 b^2;$$

hence, eliminating B between this and the preceding equation, we obtain

$$\tfrac{1}{6}\mathbf{M}_3 b + EI \tan \alpha_2 = \tfrac{1}{24}w_2 b^3 - \tfrac{1}{3}\mathbf{M}_2 b,$$

or

$$EI \tan \alpha_2 = \tfrac{1}{24}w_2 b^3 - \tfrac{1}{3}\mathbf{M}_2 b - \tfrac{1}{6}\mathbf{M}_3 b. \tag{39}$$

Similarly, for the segment A_1A_2, we get

$$-EI \tan \alpha_2 = \tfrac{1}{24}w_1 a^3 - \tfrac{1}{3}\mathbf{M}_2 a - \tfrac{1}{6}\mathbf{M}_1 a,$$

where w_1 is the load per unit of length for A_1A_2.

Hence

$$\mathbf{M}_1 a + \mathbf{M}_3 b + 2\mathbf{M}_2 (a + b) = \tfrac{1}{4}(w_1 a^3 + w_2 b^3). \tag{40}$$

This relation is called the *theorem of three moments.*

If the load be uniform over the whole space A_1A_3, the equation becomes

$$\mathbf{M}_1 a + \mathbf{M}_3 b + 2\mathbf{M}_2 (a + b) = \tfrac{1}{4}w (a^3 + b^3). \tag{41}$$

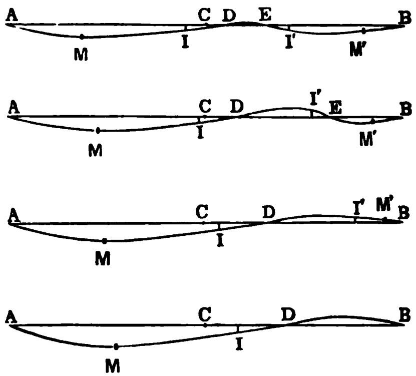
If, moreover, $\mathbf{M}_1 = \mathbf{M}_3$, we have

$$\mathbf{M}_1 + 2\mathbf{M}_2 = \tfrac{1}{4}w (a^2 + b^2 - ab). \tag{42}$$

In general, for a uniformly loaded beam supported by a number of props, whose extremities are situated in a common horizontal, we have, by (40) or (41), a number of linear equations between the several bending moments at the different props. Also, as the extreme bending moments are in general known, it is easily seen that we have sufficient equations for the determination of the bending moment at each point of support. The pressures on the props can hence be determined. A few examples will be found in the following Articles.

102. **Uniform Beam, supported by Three Props, of which Two are placed at its Extremities.**— Let A, B be the extremities of the beam, C its middle point, and D the position of the intermediate prop. Let $AD = u$.

$DB = v$, and w the load per unit of length.



The bending moments at A and B are each zero; accordingly

$$\mathbf{M} = \frac{w}{8}(u^2 + v^2 - uv), \tag{43}$$

where $\mathbf{M}$ is the bending moment at D. Next, let P_1, P_2, P_3, respectively, represent the pressures on the supports at A, D, and B, then we have

$$\mathbf{M} = \tfrac{1}{2}wu^2 - P_1 u; \tag{44}$$

$$\therefore \quad P_1 = \frac{w}{8}\left(3u + v - \frac{v^2}{u}\right). \tag{45}$$

In like manner

$$P_3 = \frac{w}{8}\left(3v + u - \frac{u^2}{v}\right).$$

Also, since $P_1 + P_2 + P_3 = w(u + v) = W$, where W denotes the whole load, we have

$$P_2 = \frac{W}{8}\left(5 + \frac{(u - v)^2}{uv}\right). \tag{46}$$

If we suppose $AD > DB$, then will P_1 be greater than P_3.

If $u = v$, we get, as in Art. 99,

$$P_1 = \tfrac{3}{16}W, \quad P_2 = \tfrac{5}{8}W, \quad P_3 = \tfrac{3}{16}W.$$

Again, if $P_3 = 0$, we have

$$u^2 = uv + 3v^2;$$

this gives $u = \frac{1}{2}v(1 + \sqrt{13}) = 2.3v$, approximately.

Hence, if $\dfrac{u}{v}$ be greater than 2.3, the pressure P_3 would become *negative*; and accordingly, in that case, it would require an additional weight applied at B to bring the beam down so as to rest on the prop there placed.

It is easy to find the equations to the curves AD and DB, respectively; for at any distance x from A we have

$$\mathbf{M}_x = \tfrac{1}{2}wx^2 - P_1x, \tag{47}$$

or

$$EI\ddot{y} = \tfrac{1}{2}wx^2 - P_1x.$$

Hence

$$EI\dot{y} = \tfrac{1}{6}wx^3 - \tfrac{1}{2}P_1x^2 + C,$$

where C is a constant;

$$\therefore \quad EIy = \tfrac{1}{24}wx^4 - \tfrac{1}{6}P_1x^3 + Cx. \tag{48}$$

Again, when $x = u$ we have $y = 0$; accordingly

$$C = \tfrac{1}{6}P_1u^2 - \tfrac{1}{24}wu^3 = \tfrac{1}{48}w\,u\,(u^2 + uv - v^2),$$

and the equation to AD becomes

$$EIy = \tfrac{1}{48}wx(u - x)\left\{\left(u + v - \frac{v^2}{u}\right)(u + x) - 2x^2\right\}. \tag{49}$$

The equation to the curve BD is obtained by an interchange of letters.

Again, if AD be the greater segment of AB when cut in extreme and mean ratio, we have

$$u + v - \frac{u^2}{v} = 0,$$

and the curve BD is represented by

$$EIy = -\tfrac{1}{24}wx^3(v - x). \tag{50}$$

In this case the slope at B vanishes.

Again, from (47), we see that the point of inflexion I on AD is given by the equation

$$P_1 = \tfrac{1}{2}wx_1,$$

or

$$4x_1 = 3u + v - \frac{v^2}{u};$$

$$\therefore \quad DI = u - x_1 = \tfrac{1}{4} \cdot \frac{u^2 + v^2 - uv}{u}.$$

Also, the point M at which the upward bending moment is a maximum is given by the equation $P_1 = wx_1$; accordingly M is the middle point of AI.

It can be shown without difficulty that the bending moment at M is always less than the bending moment at D. Accordingly, if the uniform load be increased, the beam would always rupture first at the intermediate prop.

In general, if AB be divided at M and M', so that

$$P_1 : P_2 : P_3 = AM : MM' : M'B; \qquad (51)$$

then, by (8), the shearing forces vanish at the points M and M'; and hence the corresponding upward bending moments are maxima at those points.

In the accompanying figure (see page 118) the upper curve represents the form of the beam when $CD = \frac{1}{10}AC$.

In the second figure, we have $CD = \frac{1}{5}AC$; in the third, $CD = \frac{3}{10}AC$, and in the fourth, $CD = \frac{2}{5}AC$.

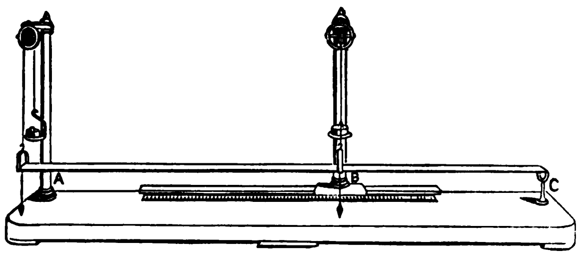
The following table contains, to the nearest integer, the calculated values of the pressures P_1, P_2, P_3, on the props; and of a_1, a_2, a_3, the corresponding angles of slope, for the different curves in the figure, where we assume the whole load to be 1000:—

$$(1)\left|\begin{array}{ll} P_1 = 216, & a_1 = 1{\cdot}53\,a_0 \\ P_2 = 630, & a_2 = \ \ {\cdot}4\,a_0 \\ P_3 = 154, & a_3 = \ \ {\cdot}54\,a_0 \end{array}\right|, \quad (2)\left|\begin{array}{ll} P_1 = 242, & a_1 = 2{\cdot}11\,a_0 \\ P_2 = 646, & a_2 = \ \ {\cdot}77\,a_0 \\ P_3 = 112, & a_3 = \ \ {\cdot}13\,a_0 \end{array}\right|,$$

$$(3)\left|\begin{array}{ll} P_1 = 264, & a_1 = 2{\cdot}74\,a_0 \\ P_2 = 674, & a_2 = 1{\cdot}1\,a_0 \\ P_3 = \ \ 62, & a_3 = -{\cdot}2\,a_0 \end{array}\right|, \quad (4)\left|\begin{array}{ll} P_1 = 284, & a_1 = 3{\cdot}42\,a_0 \\ P_2 = 720, & a_2 = 1{\cdot}34\,a_0 \\ P_3 = -4, & a_3 = -{\cdot}46\,a_0 \end{array}\right|.$$

In the above, a_0 represents the slope in the case where the intermediate prop is placed at the middle point of the beam.

103. **Apparatus for Testing the Results of the Preceding Article.**—The accompanying figure represents



a mechanical apparatus devised for the purpose of testing the results just arrived at, and so far verifying the general principles on which the theory of the bending moment depends. In the apparatus there is introduced a fixed pillar at A, and a fixed prop at C, while the intermediate pillar at B is moveable. To each pillar is attached a pulley, supporting a scale pan and a *stirrup*, which are connected by a string that passes over the pulley.

In making an experiment the central pillar is placed in any position, and a uniform flat steel bar of length AC, is made to pass through the two moveable stirrups, and to rest on a fixed stirrup at C, as represented in the figure; weights are then placed in the scale pans until the three stirrups are brought into a common horizontal line. In this manner Mr. Lilly, Assistant to the Professor of Engineering, has independently determined the weights which give the pressures on the stirrups for the different positions discussed in Art. 102. Subjoined are the results of four experimental observations for each position.

In each case after making one observation the bar was turned upside down. The same observations were made after reversing the ends of the bar.

The following are the observed results, as compared with

those calculated in Art. 102, the weight of the bar being 1271 grams:—

(1)

	P_1	P_2	P_3
	274	809	188
	273	812	186
	271	805	195
	271	804	192
av.	272	807	192
cal.	275	800	196

(2)

	P_1	P_2	P_3
	300	837	134
	305	832	134
	302	824	145
	302	827	142
av.	302	830	139
cal.	308	821	142

(3)

	P_1	P_2	P_3
	332	870	69
	335	865	71
	336	855	80
	331	857	83
av.	334	862	75
cal.	335	859	79

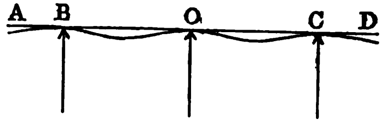
(4)

	P_1	P_2	P_3
	367	904	,,
	362	910	,,
av.	364	907	,,
cal.	361	915	-5

The close agreement of the observed to the calculated values furnishes a remarkable confirmation of the general theory of the bending moment in the case of a thin beam.

We shall next illustrate the theorem of three moments by applying it to the discussion of the best position of props when supporting a uniform and uniformly loaded beam, in order to give the maximum strength to the beam.

104. **Beam on Three Props.**—We commence with the case of uniformly loaded beam supported by three props.



It may be assumed that, in the strongest position, the intermediate prop should be placed under the middle point O of the beam, and that the other props must be equidistant from

its extremities. It can now be shown that, in order to bear the greatest load, the bending moments at B, O, and C must be all equal, as in Art. 88.

Let $\quad BO = u, \quad AB = \beta$; then $\mathbf{M}_B = \mathbf{M}_c = \frac{1}{2}w\beta^2$,

and if $\mathbf{M}_0 = \mathbf{M}_c$ we must have, by (42),

$$\mathbf{M}_0 = \mathbf{M}_c = \tfrac{1}{12}wu^2; \tag{52}$$

accordingly
$$u = \beta\sqrt{6}. \tag{53}$$

Again, if $\mathbf{M}_1$ be the bending moment at the middle point of CO, we have, as in (6),

$$\mathbf{M}_1 = \mathbf{M}_c - \tfrac{1}{8}wu^2 = -\tfrac{1}{24}wu^2 = -\tfrac{1}{2}\mathbf{M}_c;$$

we accordingly see that $\mathbf{M}_c$ is the greatest bending moment for the entire beam; and hence, if the load be uniformly increased, the beam would first give way at the props.

Again, if l be the whole length of the beam, we get

$$\mathbf{M}_c = \frac{wl^2}{8(1+\sqrt{6})^2} = \frac{Wl}{95}, \text{ q.p.}, \tag{54}$$

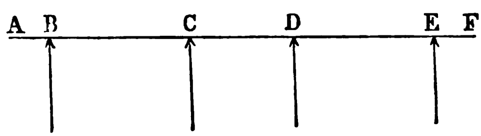
where W represents the whole load.

There is no difficulty in seeing that any alteration in the position of the props will increase the value of the *maximum* bending moment.

Again, since the neutral fibre touches AD at O, it must also touch it at C, since $\mathbf{M}_0 = \mathbf{M}_c$; accordingly in this case we may regard the beam as firmly embedded at the points B, O, C, without altering the conditions of the problem.

This also shows the length AB of the beam that should be imbedded in the masonry.

105. **Strongest Position for Four Props.**—The case



of four props admits of a similar investigation. We shall

first assume that the props are symmetrically placed relative to the middle point of the beam. Let

$$AB = \beta, \quad BC = DE = u, \quad CD = v,$$

and let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the bending moments at B and C, respectively; then by (40) we have in all cases,

$$\mathbf{M}_1 u + \mathbf{M}_2 (2u + 3v) = \tfrac{1}{4} w (u^3 + v^3).$$

Now let $\mathbf{M}_2 = \mathbf{M}_1 + X$, and we get, since $\mathbf{M}_1 = \frac{1}{2} w\beta^2$,

$$X \frac{2u + 3v}{3(u + v)} = \tfrac{1}{12} w (u^2 + v^2 - uv - 6\beta^2).$$

Again, if the segments u, v, β, are connected by the equation

$$u^2 + v^2 - uv = 6\beta^2, \tag{55}$$

we have $X = 0, \quad \text{or} \quad \mathbf{M}_1 = \mathbf{M}_2.$

Also if l be the length of the beam, we have

$$l = 2\beta + 2u + v. \tag{56}$$

It can now be shown that the strongest position of the props is that for which β is a *minimum*, while it satisfies equations (55) and (56). In this case we have, by (56), $dv = -2du$, and by (55),

$$2udu + 2vdv - udv - vdu = 0,$$

accordingly we have $4u = 5v$; (57)

hence, if $u = 5a$, we have $v = 4a$, and $\beta = a\sqrt{\frac{7}{2}}$, (58)

consequently, $\mathbf{M}_1 = \frac{1}{2} w\beta^2 = \dfrac{wl^2}{8(1 + \sqrt{14})^2} = \dfrac{Wl}{180}$, *q.p.*, (59)

where W, as before, represents the whole load on the beam.

That this furnishes a minimum solution is immediately seen.

Again, since the bending moments at D and E are in this case equal, it is immediately seen that the greatest *upward* bending moment for the portion DE of the beam is represented by

$$\tfrac{1}{8}wu^2 - \mathbf{M}_1 = \frac{w}{2}\left(\frac{u^2}{4} - \beta^2\right) = \frac{11}{28}w\beta^2. \qquad (60)$$

Hence this bending moment is less than that at the prop.

It is immediately seen that the bending moment at O, the middle point of the beam, is also less than $\mathbf{M}_1$. Consequently, when the beam is divided at its points of support as in (58) it is most liable to rupture at the points of support; and it can be easily shown that this furnishes the strongest position of the four props for supporting a uniform load.

In order completely to establish this result, suppose, the position of the end props being unaltered, the two intermediate props to assume new positions, and let

$$BC = u_1, \quad CD = u_2, \quad DE = u_3.$$

Then, by the theorem of three moments, we have

$$\mathbf{M}_1 u_1 + 2\mathbf{M}_2(u_1 + u_2) + \mathbf{M}_3 u_2 = \frac{w}{4}(u_1^3 + u_2^3).$$

Again, let $\mathbf{M}$ denote the value when $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_3$, then we have

$$\mathbf{M} = \frac{w}{12}(u_1^2 + u_2^2 - u_1 u_2); \qquad (61)$$

and it is obvious that the greatest of the three bending moments is greater than $\mathbf{M}$.

In like manner it follows that the greatest bending moment is greater than

$$\frac{w}{12}(u_2^2 + u_3^2 - u_2 u_3);$$

and consequently it is greater than half the sum of these expressions, *i.e.* greater than

$$\frac{w}{24}(u_1^2 + 2u_2^2 + u_3^2 - u_1 u_2 - u_2 u_3). \qquad (62)$$

It now can be shown that this latter expression is always greater than the value of M_1 found in (59). This follows by showing that the least value of

$$u_1^2 + u_3^2 + 2u_2^2 - u_2(u_1 + u_3)$$

is that which it has when the beam is divided as in (58).

For, let $$\phi = u_1^2 + u_3^2 + 2u_2^2 - u_2(u_1 + u_3),$$

where $$u_1 + u_2 + u_3 = l - 2\beta.$$

Here for the minimum value of ϕ we have

$$(2u_1 - u_2)\,du_1 + (4u_2 - u_1 - u_3)\,du_2 + (2u_3 - u_2)\,du_3 = 0,$$

and also $$du_1 + du_2 + du_3 = 0\,;$$
hence we get
$$2\,(u_1 - u_3)\,du_1 + (5u_2 - u_1 - 3u_3)\,du_2 = 0,$$

or $$u_1 - u_3 = 0, \quad 5u_2 - u_1 - 3u_3 = 0\,;$$

this agrees with (58).

There is no difficulty in seeing that this corresponds to a minimum value of ϕ, *i.e.* of the expression in (62).

If now the extreme props be each moved nearer to the ends of the beam, so as to diminish the bending moments at these props, it is immediately seen that the expression in (62) will be increased; so that in all cases we infer that the position of the props arrived at in (58) gives that of the greatest strength.

106. **Position of Props in General for Maximum Strength.**—The method of the two preceding Articles can be easily extended to the case of any number of props. Thus for an *odd number*, the bending moment should be the same for all the props, and hence the interval between each successive pair of props must be the same. The solution follows readily by the method adopted in Art. 104.

For an *even number* $(2m)$ of props, if the bending moments be all equal, we see immediately that the distance between the third and fourth props must be equal to that

between the first and second; also that if u_1 be the distance between the first and second, and u_2 that between the second and third, the successive intervals will be equal to u_1, or else to u_2, alternately.

Also the common bending moment for the props is represented by

$$M = \frac{w}{12}(u_1^2 + u_2^2 - u_1 u_2);$$

hence if β denote as before the projection of the beam beyond the first and last props, we have, as in (55),

$$u_1^2 + u_2^2 - u_1 u_2 = 6\beta^2; \qquad (64)$$

we have also

$$l = 2\beta + m u_1 + (m-1) u_2. \qquad (65)$$

The result can be readily arrived at by differentiation, as in Art. 104. It can also be shown easily by ordinary Algebra, as follows; for when M is a minimum, β must also be a minimum; this gives

$$\lambda = \frac{u_1^2 + u_2^2 - u_1 u_2}{(u_1 + a u_2)^2}, \text{ a minimum, where } a = \frac{m-1}{m}.$$

Solving this as a quadratic equation for $\frac{u_1}{u_2}$ we have

$$(1-\lambda) u_1 - (\tfrac{1}{2} + \lambda a) u_2 = u_2 \sqrt{\lambda(1 + a + a^2) - \tfrac{3}{4}}; \qquad (66)$$

consequently the least value of λ, consistent with *real values* for u_1 and u_2, is given by the equation

$$\lambda = \frac{3}{4(1 + a + a^2)}.$$

Substituting in (66) we get, after an easy reduction,

$$(1 + 2a) u_1 = (2 + a) u_2,$$

or

$$\frac{u_1}{u_2} = \frac{3m-1}{3m-2}. \qquad (67)$$

Now let $u_1 = (3m-1)\alpha, \quad u_2 = (3m-2)\alpha,$ and we get

$$\beta^2 = \tfrac{1}{2}\alpha^2 (3m^2 - 3m + 1),$$

hence $l = \alpha(1 + \sqrt{6m(m-1)+2})\sqrt{6m(m-1)+2}$;

consequently the minimum value of the bending moment $\mathbf{M}$ is given by

$$\mathbf{M} = \tfrac{1}{2}w\beta^2 = \frac{Wl}{8\left(1+\sqrt{6m(m-1)+2}\right)^2}; \qquad (68)$$

where, as before, W represents the whole load.

It is easily seen that in this case $\mathbf{M}$ is greater, regardless of sign, than the bending moment at any other point on the beam. It can also be shown, as in Art. 105, that if the beam be divided by its points of support in any other manner than that arrived at above, one at least of the bending moments at the props will be *greater* than the value of $\mathbf{M}$ as determined in (68).

107. **Consideration of the Preceding Theory.**— This chapter has been to a great degree written for the use of the practical student, and the investigations contained in it have been based on the two assumptions stated in Art. 91, viz. that the cross-sections of the beam still remain plane after flexure, and also that the stresses for the fibres which are parallel to the axis of the beam follow Hooke's law; in other words, I have adopted what is now generally known as the Bernoulli-Eulerian theory. That the foregoing hypotheses are, in general, untenable has been shown by Saint-Venant and others, but the most complete investigation of the problem is that which has been recently given by Professor Pearson.*

In fact it can be shown that the hypotheses in question only hold when the bending moment $\mathbf{M}$ is a linear function of x, the distance from the origin measured along the beam. This has been readily shown by Professor Pearson. In the following demonstration we take the neutral axis as that of x, the axis of y being vertical; then, reverting to the notation for stress and strain adopted in the earlier part of the book, equations (14) and (15) may be written

$$A = Ea = E\frac{du}{dx} = \frac{\mathbf{M}}{I}y, \qquad (69)$$

* "On the flexure of heavy beams subjected to continuous systems of load," Quarterly Journal of Mathematics, 1889, pp. 63–110.

in which, for uniform beams, the bending moment $\mathbf{M}$ may be taken as a function of x solely.

Hence
$$E\frac{d^2u}{dx^2} = \frac{y}{I}\frac{d\mathbf{M}}{dx}, \tag{70}$$
and also
$$Eu = \frac{y}{I}\int \mathbf{M}dx + \chi, \tag{71}$$

where χ is an arbitrary function of y and z.

In investigating the legitimacy of these equations we may regard the beam as isotropic.

Accordingly the equations of Art. 63 give

$$A = \lambda\Delta + 2\mu a\ ;\ \therefore\ \Delta = \frac{E-2\mu}{\lambda}a = \frac{E-2\mu}{\lambda}\frac{du}{dx}. \tag{72}$$

Also, by Art. 68, we have, since the load acts perpendicular to the beam,

$$(\lambda+\mu)\frac{d\Delta}{dx} + \mu\nabla^2 u = 0,$$

or

$$\left(\frac{\lambda+\mu}{\lambda}E - \mu - \frac{2\mu^2}{\lambda}\right)\frac{d^2u}{dx^2} + \mu\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 0.$$

By (70) and (71) this may be written in the form

$$Ky\frac{d\mathbf{M}}{dx} + \mu\left(\frac{d^2\chi}{dy^2} + \frac{d^2\chi}{dz^2}\right) = 0, \tag{73}$$

where K is a constant.

Now this equation cannot hold good for all points on the beam unless

$$\frac{d\mathbf{M}}{dx} = \text{const.},$$

i. e. unless
$$\mathbf{M} = cx + c'.$$

Accordingly, equations (69) hold only when the bending moment is due to an *isolated load*, and consequently all the results are inexact for the case of continuous loads.

The practical question remains as to how far the received theory, that the bending moment at any section of the beam is proportional to the curvature, is an *approximation* to the truth.

With this object in view, Professor Pearson has, adopting the *semi-inverse method* of Saint-Venant, fully discussed the case of a heavy beam of circular cross-section subject to a surface load, which is perpendicular to the axis of the beam.

His investigation shows "*that, in the case of a beam continuously loaded, the traction in any fibre does not theoretically vary either* (*a*) *as the bending moment, or* (*b*) *as the distance from a neutral axis.*"

The general conclusions arrived at by Professor Pearson are stated by him as follows :—

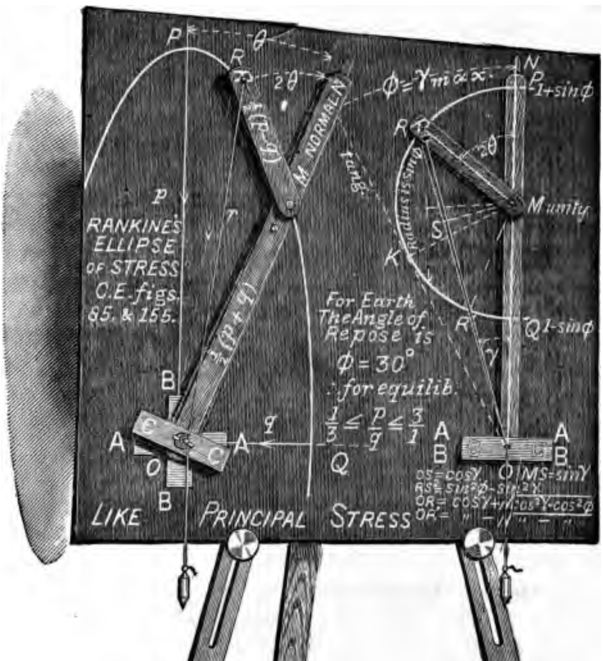
(1) All the theoretical assumptions on which the Bernoulli-Eulerian theory of beams is based are in the exact sense false. The cross-sections are distorted; there is no true "neutral axis," and the curvature does not vary as the bending moment.

(2) But, if the true theory of the beam be worked out, then the results of the Bernoulli-Eulerian theory will be found to give fairly approximate formulæ for the stress and strain of such beams as occur in ordinary practice (i.e. beams whose diameter is $\frac{1}{10}$th or less of their length).

We shall conclude with a brief account of Professor Alexander's method of practically applying Rankine's "Stress Ellipse," which has been already treated of in Art. 55.

108. **Model illustrating Rankine's "Ellipse of Stress."**—The accompanying diagram represents the Model in the Engineering Laboratory of Trinity College, Dublin, devised by Professors Alexander and Thomson for illustrating Rankine's Method of the Ellipse of Stress. On the left hand of the diagram *ON* is a T square, pivoted to the black board at *O*, while *MR* is a pointer pivoted to the blade of the T square at *M*. On the pivot *M* a wheel is fixed at the back of the blade, and round the wheel an endless chain is wrapped, which also wraps round a wheel fixed to the board at *O*. The wheel at *O* is of a diameter double that of the wheel

at *M*. Hence, when the blade of the T square is turned to the right, the pointer *MR* automatically turns to the left, so that the angle *RMN* is always equal to 2θ, where $PON = \theta$; or, in other words, the bisector of *RMN* is always parallel to *OP*. A string, fastened to the pointer at *R*, passes through a swivel ring at *O*, and is kept tight by a plummet. It is easily seen that the point *R* describes an ellipse.



Again *CC*, the head stock of the T square, represents the trace on the black-board of any plane through *O* normal to the board; and the vector *RO* of the ellipse represents the stress on the plane *CC*, in intensity and direction, by Art. 55. Also the angle $PON = \theta$ gives the position of the plane *CC* relative to the plane of the greater principal stress *AA*, while $RON = \gamma$ is the obliquity of the stress upon *CC*.

The model shows clearly the interesting positions of the plane CC; thus CC may be turned to coincide with AA, when the string RO will be found to be normal to CC, and to be of a maximum length. On the other hand, if CC be turned to coincide with BB, the string RO is again normal to CC, but of a minimum length. Again, when CC is so placed that the angle RMO is a right angle, the component of RO parallel to CC is a maximum; and lastly, if it be turned till MRO is a right angle, then ROM the obliquity of the string is a maximum.

The auxiliary figure on the right of the diagram is for solving the general problem of uniplanar stress at a point, viz. given the stress in intensity and obliquity for two positions of CC, to find the stress for any third required position of CC. On the auxiliary figure, the T squares for all positions of CC are superimposed upon each other, and are represented by one T square fixed to the board, and only the pointer MR turns.

In solving questions on the stability of earthworks, some linear dimension upon the auxiliary figure represents the known weight of a column of earth, while another dimension represents the required stress on a retaining wall, or on a foundation, &c. Generally two angular quantities also are known, such as $KON = \phi$, the maximum value of γ, this being the steepest possible slope of the loose earth, while $RON = \gamma$ may be taken as the known slope of the surface of the loose earth.

On the other side of the board, another T square is similarly pivoted to the same pin O, the only difference being that OM is shorter than the pointer MR. It represents the case of Unlike Principal Stresses, and is useful in demonstrating the composition of stresses such as that of thrust and bending on a pillar, or of bending and twisting on a crank pin.

By means of the slots on the easel the board can be placed so that any desired vector of the ellipse may be vertical.

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